This paper was written Sept. 2000 and published 2003 on my website (2017 partly corrected and updated):
www.ucke.de/christian/physik/ftp/lectures/Wobb3e.pdf

The Two-Disc-Roller – a Combination of Physics, Art and Mathematics

Christoph Engelhardt, *~~Hans-Pfann-Str. 74, 81825~~ München, Germany*
Christian Ucke*, Faculty of Physics, Techn. Univ. of Munich, Germany*

1 Introduction

If a homogeneous sphere rolls off on a tilted plane the distance between the center of gravity and the plane remains constant. The same phenomenon applies to revolving cylinders. There are other objects of this behavior, for example two-ellipse like discs which are cut out from cylinders (Fig. 1 left-hand side). The path of the center of gravity describes a straight line in all these examples.

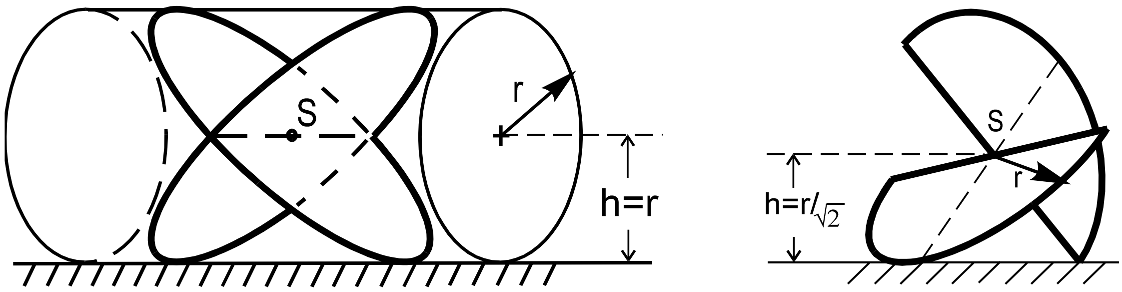

Fig. 1. The centre of gravity S of two ellipse-like discs cut out from a revolving cylinder remains in a constant distance from the plane while rolling off on this plane. The same applies to two half-circle-discs interlocked perpendicularly.

The next example consists of two – mathematically idealized infinitely thin – half-circle-discs, which are interlocked perpendicularly to each other at the original center (Figure 1, right-hand side). If this object rolls down a slightly tilted plane, the distance between the center of gravity and the plane remains constant. The path of the center of gravity is no longer a straight line but is more like a serpentine. Exactly regarded, this line is composed of circular arches, as it is shown later on. Because of this movement, such objects are called in English **wobblers** (from the verb: to wobble)

According to this principle the Swiss artist **Rolf Hergert** [1] has created an object which he calls **Go-On** (Fig. 2). It is made of transparent LISA-plastics, which is often used in the decoration branch. Fluorescent molecules are incorporated in the plastics collecting the light and letting it emerge at the edges.

Rolling down a plane, such a wobbler touches this always in two points. By connecting all corresponding bearing-surface points to each other a convex hull is obtained, also named connecting developable (Fig.3). The German artist **Karsten Hein** had such a wooden body protected for himself under the name **Quirr** (Fig.4).

Fig.2: The Swiss artist Rolf Hergert has created a two-disc-roller named Go-On. It is exactly based on the principle that two semi-circles are interlocked perpendicularly one to another.
(diameter discs ~2.5cm)

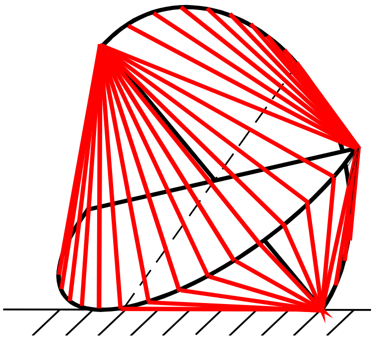

Fig. 3: By connecting all corresponding bearing-surface points of the wobbler to each other the so-called connecting developable are obtained.

Fig.4: The German artist Karsten Hein has created an object called Quirr which is based on the principle described in Fig. 3 [2].
(*size ~8cm)*

One can take now a further step and wonder what will happen if two entire circular discs are interlocked perpendicularly to each other. This can be made practically very easily by cutting radial slits into the circular discs. The result is displayed in Figure 5.

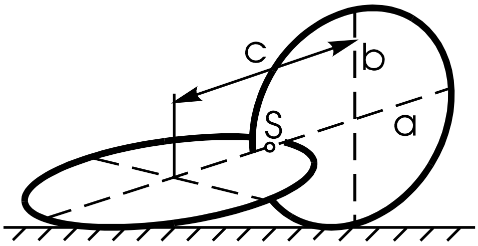

Fig 5: Two entire circular discs interlocked perpendicularly one to another result in a further sort of wobblers. Also using ellipse-like discs, (half-axes a, b) wobblers can be constructed whose center of gravity distance from the rolling-off plane remains constant.

Using such a construction, – while the wobbler rolling off on a plane – , the distance between the center of gravity and the plane remains exactly constant, if the distance between the centers of the two circular discs fulfills the condition indicated in Figure 5. (for the mathematical derivation see later on)

There are so-called beer-mats that are excellently suitable for self-construction of wobblers. They are easily available in some countries like Germany or Great Britain, mostly circular-like, sometimes also ellipse-like. They can be easily processed with a knife and some glue. Figure 6 shows an example with ellipse-like beer-mats. The authors forwarded this sample to the corresponding brewery in order to get some more mats. Not only obtained we several hundreds of items but furthermore two boxes of beer „to facilitate the science-work“. From which it can be seen that such an investigation „can be worthwhile“.

Fig. 6: A wobbler can be constructed of ellipse-like beer-mats.
(big ellipse axis ≈ 11cm)

The principle of rollers made of two entire circular discs is realized in miscellaneous toys. We mention here the Finnish children's toy **Ensihammas** (Fig. 7)[3]. The shaking movement seems to fascinate children, too.

Using two parts of a construction toy, called **RONDI** (Fig. 8), one can combine two discs immediately to a wobbler. The distance condition in Fig.5 is well fulfilled. We asked the construction enterprise [4] about this property. It resulted that it was a coincidence.

While rolling off these two-disc-rollers always touch a plane exactly in two points, too. By connecting the contact-points of the roller to each other one obtains the connecting developable (see Fig. 11). This is a aesthetic looking body. The English artist **Rick Flowerday** [6] converted this idea to a small toy (Fig.9). Flowerday also published this kind of connecting developable [7]. In his publication he discusses among other things the question of the influence of a finitely thick disc concerning two rollers of two entire circular discs. Other yet unsolved problems, such as the volume and the surface of the connecting developable are discussed.

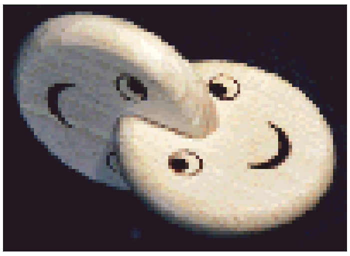

Fig.7: The Finnish children's toy Ensihammas is made of two thick circular discs.
(diameter discs ≈ 6cm)

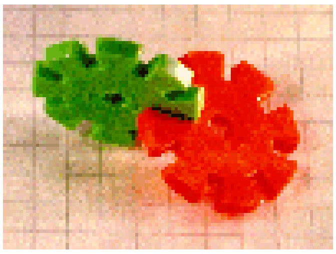

Fig. 8: With the construction toy RONDI a two-disc-roller can be put together.
(diameter discs ≈ 2.5cm)

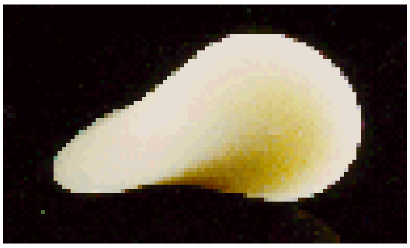

Fig. 9: The English artist Rick Flowerday took the connecting developable of a roller of two entire circular discs as a pattern for a wobbler.
(diameter discs ≈ 3cm)

2 Mathematical Derivation

In a publication of the year 1966 the Two-Circle-Roller was treated for the first time, but only discussing the case of the circular discs [8]. In the following discussion a derivation is given generalized for ellipses. Further generalizations are possible [5].

For the two congruent ellipse edges, being interlocked perpendicularly to each other, in a *body-fixed, cartesian coordinate system* of the two-disc-roller we choose a parameter estimation as follows (compare to Fig.: 10).

(1) Ellipse 1: $$x = a \cdot \sin\varphi + \frac{c}{2} \qquad y = b \cdot \cos\varphi \qquad z = 0$$

(2) Ellipse 2: $$x = a \cdot \sin\psi - \frac{c}{2} \qquad y = 0 \qquad z = b \cdot \cos\psi \, .$$

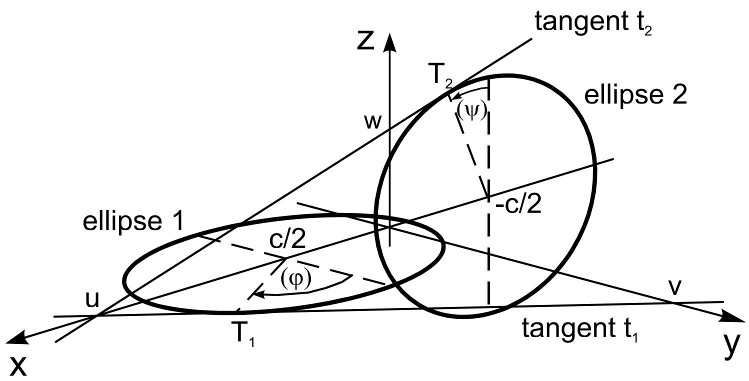

Fig. 10: Sketch concerning the position of the ellipses and a suitable coordinate system. In the case of circles the parameters φ and ψ can be attached to angles at the circular centre points.

a, *b* correspond to the half-axes, c is the distance between the ellipse centers. The ellipse 1 lies on the *x/y*-plane, its center on the *x*-axis at +*c/*2, the ellipse 2 is situated on the *x/z*-plane, is center on the x-axis at -c/2. Because of this position symmetry the center of gravity of the ellipse-combination (1) and (2) lies in the coordinate origin.

A tangent t_1 touching the ellipse 1 in the point $T_1(\varphi)$ can be represented as

(3) $$b\cdot x\cdot\sin\varphi \;+\; a\cdot y\cdot\cos\varphi \;= a\cdot b+\frac{b\cdot c}{2}\cdot\sin\varphi$$

The axes sections of this tangent results from this to

(4) $$u=\frac{a}{\sin\varphi}+\frac{c}{2} \quad ; \quad v=\frac{b\cdot u}{a}\cdot\tan\varphi$$

Likewise one can find the axes section of a tangent t_2, touching the ellipse 2 in the point $T_2(\psi)$, and which goes through the axis section *u*:

(5) $$u=\frac{a}{\sin\psi}-\frac{c}{2} \quad ; \quad w=\frac{b\cdot u}{a}\cdot\tan\psi\ .$$

From this the coupling condition for φ and ψ results in:

(6) $$\frac{a}{\sin\psi}-\frac{a}{\sin\varphi}=c\ .$$

The tangents t_1 and t_2 lay with the two points T_1 and T_2 on a common tangential plane τ on to the both disc edges. This tangential plane is in the laboratory system, in which the rolling action of the two-disc-roller is observed, exactly the rolling-off plane. The shape of the axis section of τ can be written in general as

(7) $$\tau:\ \ \frac{x}{u}+\frac{y}{v}+\frac{z}{w}=1\,.$$

With (4) and (5) the tangential plane can be described as follows:

(8) $$\tau:\ \ b\cdot x+a\cdot y\cdot\cot\varphi+a\cdot z\cdot\cot\psi=b\cdot u$$

The distance *h* of this plane τ from the origin of the coordinate system also represents the distance between the center of gravity and the tangential-(roll off)-plane τ :

(9) $$h={bu}\big/{\sqrt{q}} \qquad \text{mit} \quad \mathrm{q}=b^2+a^2\cot^2\varphi+a^2\cot^2\psi$$

ψ can be eliminated by means of the coupling condition (6) so that the following result is finally obtained:

(10) $$h = b \cdot (a + \frac{c}{2} \cdot \sin\varphi) \Big/ \sqrt{Q} \quad \text{with} \quad Q = (b^2 + c^2 - 2a^2)\sin^2\varphi + 2ac\sin\varphi + 2a^2$$

The distance *h* therefore depends in general on the parameter φ, and so it is not necessarily constant. The values *a, b, c*, are now --- as far as possible --- to be put in relation to each other in such way that *h* no longer depends on φ. For this it is necessary that the discriminant of the polynom Q being square in $\sin\varphi$ is eliminated which is exactly the case for

(11) $$c^2 = 4a^2 - 2b^2.$$

This results in $Q = \left[\sqrt{2a^2 - b^2}\sin\varphi + \sqrt{2}a\right]^2$ and it really follows for *h* the constant value

(12) $$h = b \big/ \sqrt{2}\ .$$

The condition (11) results necessarily in the relation $a \geq b/\sqrt{2}$, concerning the half-axes *a, b* of the ellipse edges of a two-disc-roller having a constant center of gravity distance from a flat bearing-surface area. $a = b/\sqrt{2}$ is obtained for the limiting case $c = 0$. That is the case corresponding to Figure 1; in this case the disc edges are just the two sectional ellipses of two turning cylinders whose axes cut each other perpendicularly.

By connecting the bearing-points of the two-disc-roller with straight line pieces one obtains the respective connecting developable (Fig.11). If the subsequent touching-points are marked on a piece of paper (see Fig. 21) the connecting developable can be cut out and stuck together, which results in a very aesthetic body. Using (3) very illustrative thread models of the connecting developable can be made by connecting some bearing-points to each other, which are calculated according to (3), for example using a two-disc-roller made of thin wooden discs.

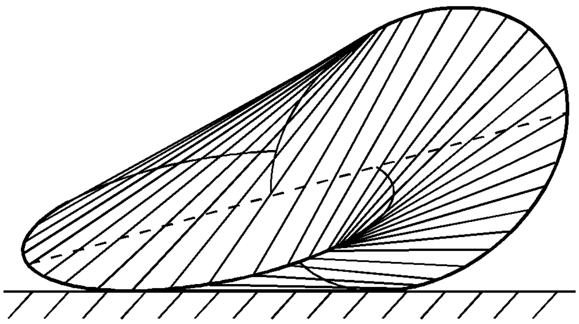

Fig. 11: By connecting the respective bearing-points of the two-disc-roller to each other an aesthetically looking body, called 'connecting developable's by mathematicians, is obtained.

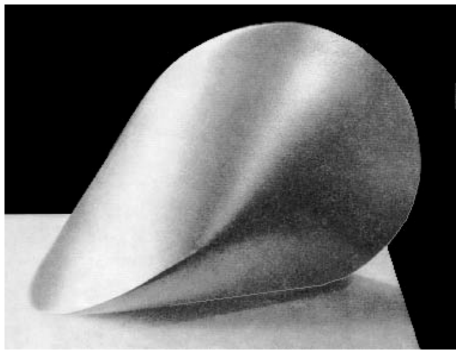

Fig. 12: The so-called Oloid looks like the two-disc-roller in a confusing way; however, it has a completely different background.

At first glance the Oloid (Fig. 12) is a completely identically looking body. Originally, it was found in a complicated way as the convex hull in the so-called upside-down-turnable cube by Paul Schatz [9].

It can be more easily constructed by employing the condition that the distance between the centers of the producing circular discs of the Oloid are exactly as long as the diameter of a circle. As this distance does not correspond to the condition (11) for a constant center-of-gravity distance, the Oloid shakes there and back after being pushed slightly. Being pushed more strongly it rolls indeed fairly easily over a plane, the reason of which is that the center of gravity varies in hight only very little (for this $c = a = b = r$ is set in (10)). Using the figures (1), (2) of both disc-edges it can be shown that, concerning the Oloid ($c = a = b = r$) --- contrary to the two-disc-roller produced of general ellipses ($c^2 = 4a^2 - 2b^2$) --- the connecting lines of the bearing points have always the same length ($= r \cdot \sqrt{3}$).

The Oloid is also used for technical applications. Special mixing-machines are constructed using such bodies [10].

3 Center of gravity path of the two-disc-roller

Another problem is the obviously wavy-line-like path of the center of gravity of the two-disc-roller by using a pair of congruent ellipses rolling off on a plane. In order to treat this question we make the following considerations:

The sphere described around the origin by the radius $h = b/\sqrt{2}$

(13) $$\Sigma: \quad x^2 + y^2 + z^2 = \frac{b^2}{2}$$

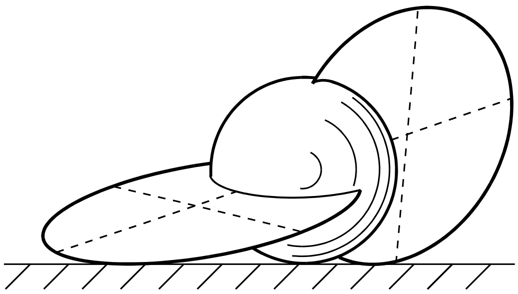

Fig.13: The drawn-in sphere touches the plane while the two-disc-roller rolls off on the bearing-plane. We call it the touching sphere.

touches each of the planes τ(7) in a specific point T(X , Y , Z). That's why the connecting developable of the two-disc-roller enclosed by the planes τ describe the sphere along a specific space curve *l*.

Comparing the coefficients of the equivalent representations

(14) $$\tau: \quad X \cdot x + Y \cdot y + Z \cdot z = \frac{b^2}{2}$$

with (8) result in the coordinates of T using (4) and (5) to

(15) $$X = \frac{b^2}{2u} \;,\; Y = \frac{b^2}{2u} \cdot \cot\varphi \;,\; Z = \frac{b^2}{2u} \cdot \cot\psi \quad \text{with} \quad u = \frac{a}{\sin\varphi} + \frac{c}{2} = \frac{a}{\sin\psi} - \frac{c}{2}$$

This is yet a parameter representation of the touching line *l* – by means of the coupling condition (6) selectively to write in φ or ψ. The elimination of φ and ψ in (15) shows that *l* takes a course on the two congruent *elliptic* cylinders according to

(16) $$\frac{(X+\frac{c}{2})^2}{a^2}+\frac{2Y^2}{a^2}=1 \quad \text{and} \quad \frac{(X-\frac{c}{2})^2}{a^2}+\frac{2Z^2}{a^2}=1 \quad \text{with} \quad c=\sqrt{4a^2-2b^2}$$

Using (13) and (16) a coordinate representation of the touching line *l* is finally produced in the body-fixed system of the disc-pair. This results further in a curve to cut the sphere Σ with the *hyperboloid paraboloid* Λ :

(17) $$l: \qquad \Sigma: \; x^2+y^2+z^2=\frac{b^2}{2} \quad , \quad \Lambda: \; c\cdot x+y^2-z^2=0$$

The touching-line *l* is therefore an algebraic space curve of the *4th order*. *l* is given as the cutting curve between the connecting developable and Λ likewise. According to the appearance one could call *l* a „*tennis ball curve*„ (Fig.14a,b).

While the roller rolls off on a plane the sphere Σ rolls along this curve *l* on the support. The impression trace produced therewith of *l* on the plane is translationally congruent to the center of gravity path of the roller. The calculation of this trace --- at least numerically --- surpasses the frame of this work. Yet without calculation several qualitative assessments can be made about the center of gravity path.

For $0<c<\infty$ --- according to (11) equivalent to $a>b/\sqrt{2}$ --- one obtains the „*tennis ball curve*„ (Fig.14). The center of gravity path is then – based on the symmetry of *l* – a periodical, sinus-like curve, whose *period* an *amplitude* therefore depends on *c*, and the half-axes *a* and *b* of the ellipse-discs according to (11).

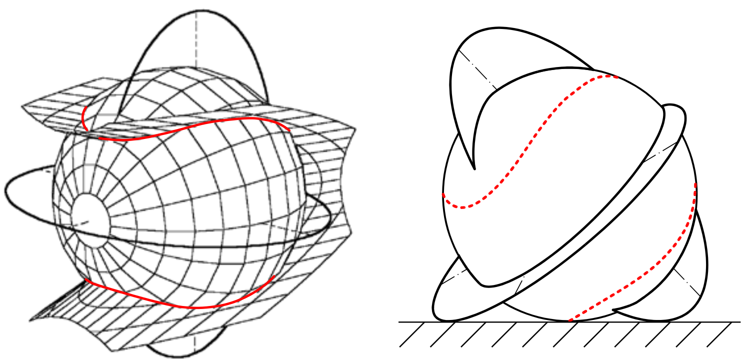

Fig.14a, b: While the two-disc-roller rolls off on a plane the touching curve rolls along a ‘tennis ball curve’, here represented by the cutting curve between the touching sphere an a hyperbolic paraboloid. For the half-axes-relation of the ellipse-discs a = 0.72·b has been chosen in this figure. Thus the central distance results in c ≈ 0.27·b.

For the limiting case of $c\to\infty$ the hyperboloid paraboloid Λ in the body-fixed system degenerates to the *y/z*-plane ($x=0$). *l* is then a circle described around the origin with the radius $r=b/\sqrt{2}$ in the *y/z*-plane. This circle produces a straight line as trace while the sphere Σ rolls off on a plane. Thus the center of gravity trace approaches a *straight* line for ellipse-discs with *a* >> *b*.

For *c* = 0, therefore $a=b/\sqrt{2}$, Λ degenerates to the pair of planes | *y* | = | *z* | which are perpendicular to each other. The space curve *l* thus disintegrates to a pair of congruent circles (radius a) lying in this pair of planes around the origin (Fig.15). The center of gravity course is therefore made up of *straight line pieces* of the length πa. The center of gravity of a two-disc-roller of this kind would naturally follow a straight line after being pushed as a result of its inertia. Mathematically, also a curve produced with straight line pieces of the length πa being perpendicular to each other would be conceivable.

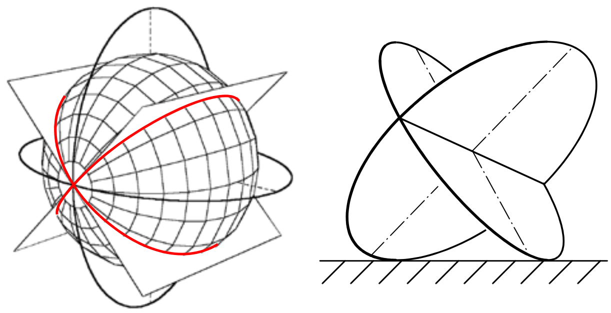

Fig.15a, b: For the limiting case c = 0, that is to say that the ellipses centers fall together having the half axis-relation $a = b/\sqrt{2}$ *, the tennis ball curve degenerates to two great circles of the touching sphere being perpendicular to each other.*

4 The half-ellipse-roller

Another two-disc-roller can be constructed following the principle of the ellipse-roller based on a pair of congruent *half-ellipses* (half-axes a, b) (Fig. 16).

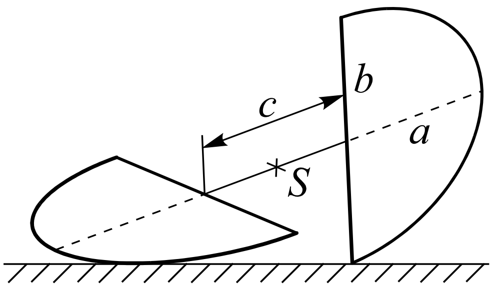

Fig.16: Two congruent half-ellipse-discs interlocked perpendicularly in a defined distance c have a constant center of gravity distance from the plane on which they lie.

With regard to the half-ellipse-roller, of which the central distance c between the ellipse centers corresponds to the equation

(18) $$c^2 = 2a^2 - 2b^2 \quad \text{with} \quad a \geq b$$

The (geometrical) center of gravity retains the constant distance $h = b/\sqrt{2}$ from a plane, while rolling off on this, --- just like the ellipse roller ---. The mathematical derivation of the condition (18) be only outlined as follows; it is developed in an analogical way to the calculation in part 2.

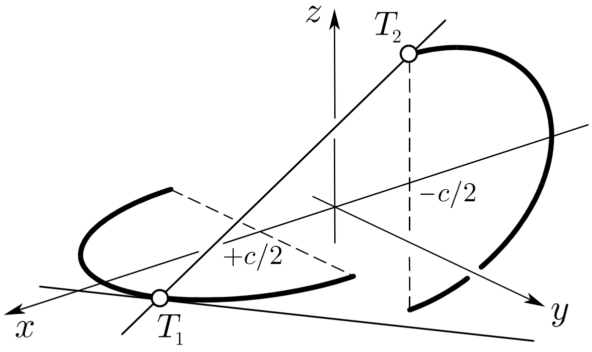

Fig. 17: Representation of the half-ellipse-roller in a body-fixed cartesian coordinate system.

The position and the parameter representation of the half-ellipses and their half-axes *a* and *b* in the body-fixed cartesian coordinate system is defined analogically to the *ellipse*-roller (‘*Wobbler*’) (Fig. 17); this construction’s center of gravity is thus situated in the coordinate origin. Contrary to the ellipse-roller it must be observed that the position of the half-ellipse 1 is chosen as follows: The half-ellipse 1 positioned in the *x/y*-plane (the ellipse’s center at $x = +c/2$) is opened in the direction oft the *negative x*-axis, the half-ellipse 2 positioned in the *x/z*-plane (the ellipse’s center at $x = -c/2$) in the direction of the *positive x*-axis.

The geometrical relation between both bearing points and the roll-off plane τ is represented as follows: One of the two discs (e.g. the half-ellipse 1) touches the plane in the point T_1 always *tangentially* on the ellipse edge, that is to say the tangent t_1 to the half-ellipse 1 in the point T_1 lies in τ. The other bearing-point T_2 is always one of both *angles* of the each other half-ellipse-disc (here the half-ellipse 2). By that the half-ellipse 2 doesn’t touch the plane τ in the point T_2 tangentially (Fig. 17). Now the straight line t_1 and the second bearing point T_2 determine the bearing plane τ clearly. By means of an appropriate parametrization of the half-ellipse 1 as in (1) a representation of the plane τ can be found analogically to (8) and be obtained its distance *h* from the coordinate origin. The demand for the constancy of *h*, i.e. its independence of the parametrization of the half-ellipse 1 results necessarily in the relation (18) for the half-axes *a, b* and the central distance *c*.

Contrary to the ellipse-roller the *sign* of the center distance *c* is essential in the mathematical construction for the geometrical shape of a half-ellipse-roller. This means that negative *c* are also permissible in the *body-fixed cartesian coordinate system*; thus the discs also can be shifted *into each other* to the center distance |*c*| (Fig.18). So one obtains particularly interesting two-disc-rollers, that show a rolling behavior different from that based on the roller for *positive c* (this can be best seen using models made of paperboard according to (18) as in Fig. 18).

Fig. 18a, b, c: The half-ellipse-roller for negative central distances c, here as the cases $a/b=\sqrt{8/7}$, $a/b=\sqrt{2}$ *, (made of perfectly fitting elliptical beer-mug-stands), a/b =* 3 *(from the right to the left).*

The connecting developable of the *half-ellipse*-roller can be analytically described in the whole as compared to the *ellipse*-roller after an extensive elementary-geometrical assessment: They are composed of congruent *pieces of rotary cone planes* (Fig.3 for the case *a* = *b*, with (18) it follows then *c* = 0, Fig.19 for *c* = +*a*). Therefore they can be unrolled in the plane as *circular-disc-sectors* set to each other (Fig.21). The particular case for *half-circles* (*c* = 0) results in congruent *circular-disc-sectors* in an unrolled shape set to each other with the opening angle $\Psi=\pi/\sqrt{2}$, thus about 127.3° (Fig. 21) [11; new link 2017]

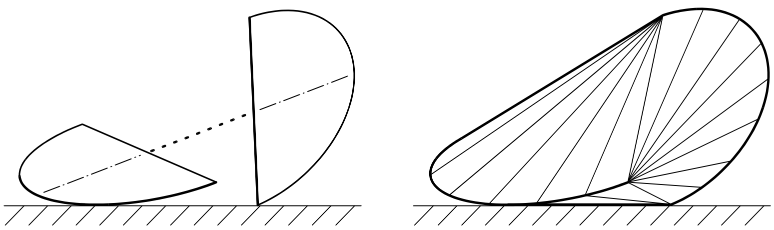

Fig. 19a, b: The connecting developable of a half-ellipse-roller is composed of pieces of rotary cone planes.

The plane path of the geometrical center of gravity can be described ‘more easily’, too. The touching line *l* of the connecting developable (here pieces of a rotary cone) with the incorporated sphere Σ (radius $h=b/\sqrt{2}$, geometrical center in the center of gravity) is made of congruent *circular arcs* of the sphere Σ and has the same symmetry characteristics such as the ‘*tennis ball curve’* with regard to the *ellipse*-roller. Thus the rolling-off of *l* into the plane delivers a *wave-like* center of gravity path which is composed of congruent *circular arcs* in *almost* all cases (Fig. 21).

The curve described on a regular tennis ball (radius *h* = 3.2 cm) (the „*tennis ball curve*„) is similar to the touching-line *l* of a half ellipse-roller in an amazingly exact way. Extensive elementary geometrical evaluations show that in this case the half-axes-relation is $a/b\approx 1.008$,i.e. that nearly half-circles with the radius $a\approx b=\sqrt{2}h\approx 4.52cm$ are obtained as disc-edges (Fig. 20). According to (18) the relating - in this case *negative* –center distance results in $c\approx -0.25h\approx 0.175a$. If such two half-circles are glued to a regular tennis ball resulting in a half-ellipse-roller including the center-of-gravity sphere □ and the touching curve *l* (*the* tennis ball curve) as shown in Fig. 20, the tennis ball touches the bearing plane just along the tennis ball curve *l* while the two-disc-roller rolls off on this flat bearing plane.

In the end the paths of the two-disc-rollers projected on the plane, and made of half-circles or entire circles respectively including the corresponding touching lines of the rollers are represented. (Fig. 21).

If one enlarges this drawing on a copy machine and cuts out the shape along the touching lines the connecting developable of the two-disc-roller can joined from this by some manual skill, as it can be seen in Fig. 3 and 11.

Although the curves look quite similar to each other the left-hand path can be easily calculated, while the right-hand path has not yet been quantitatively calculated by now [but see 12 and 13]. That is only one of the problems being yet unsolved and which we find worthwhile to treat in the future.

Coming up first from quite elementary set of questions interesting and profound connections of physical, mathematical and artistic considerations have resulted from that in this way.

Fig. 20: The tennis ball-two-disc-roller of half-ellipse-discs.

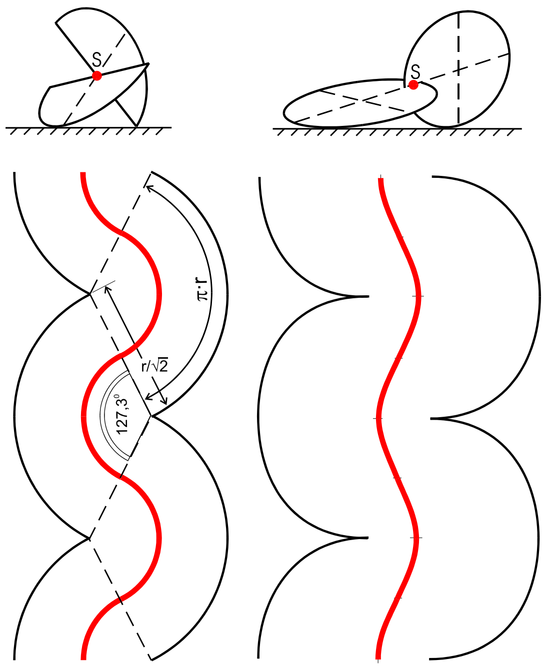

Fig.21: The paths of the centers of gravity– including the rollers made of half-circles or of two entire circles respectively – rolled off on the bearing plane.

References:

[1] Obtainable from the Swiss Gallery **AHA** (Spiegelgasse 14, 8001 Zuerich, Switzerland) no longer available

[2] Fa. **LUDUS** (Hans-Scharoun-Weg 3b, 22844 Norderstedt, Germany) no longer available

[3] **AARIKKA OY** (Nokiantie 2-4C, 00510 Helsinki, Finland). no longer available

[4] Fa. **SIMEX** (Gablonzer Str. 2-4, 76185 Karlsruhe, Germany) no longer available

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[13] see also the video: **2 Circle Roller**; www.youtube.com/watch?v=AQwVfc1vxTk

Christian Ucke: ucke@tum.de