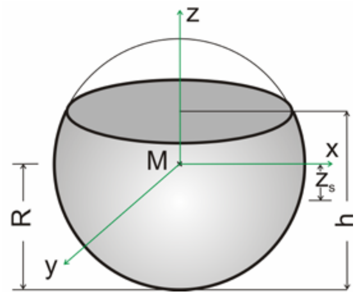
Reflections on the drinking bowl 'Balance'

1) *Is the bowl made of massive stainless steel?*
No: mass of bowl m = 153g (there are also versions with slightly different masses).
The bowl displaces a volume of V = $75cm^{3.}$
Therefore, the density would be $\rho = m/V \approx 2gcm^{-3}$. Stainless steel, however, has 7.9 to $8.0gcm^{-3}$

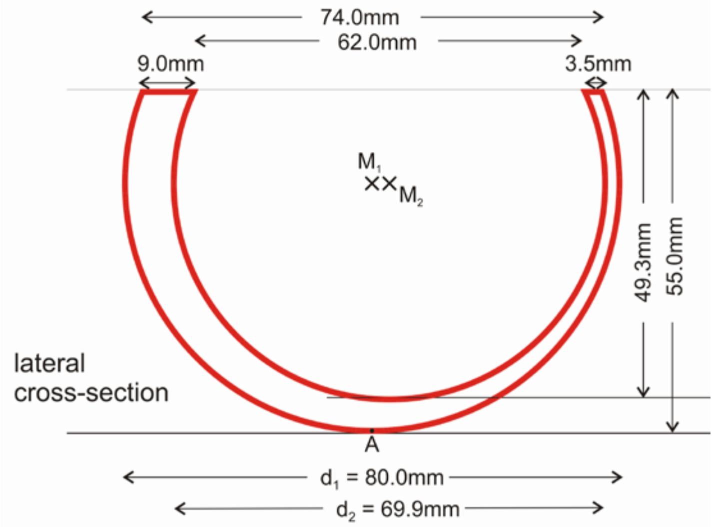
Additional question: Where would the centre of mass be if the bowl were really made of massive stainless steel? (This would be very expensive and also not function in the way intended!)
To which side would the bowl incline and how much?
The answer is somewhat more complicated!
(but possible with the calculations under point 5)



What is the thickness of the stainless steel sheet under the precondition that all the parts of the sheet have a uniform thickness and that the parts are spherical caps?

Surface of a spherical cap with the height h: $O = 2Rh\pi$

The measurements were made with a sliding calliper and are shown in the figure on the right.
There are errors in measurement of about ±0.5mm.
With the vector-orientated program CorelDraw, the inner circle was drawn through the inner bottom pole and the two inner upper border points. A calculation 'by feet' would be relatively exhausting.



For the surface of the outer spherical cap:
$O_1 = 2 \cdot 3.93 \cdot 5.43 \cdot \pi = 134.1cm^2$

surface of the inner spherical cap:

$O_1 = 2 \cdot 3.57 \cdot 4.93 \cdot \pi = 110.6cm^2$

The upper eccentric circular ring has a surface of

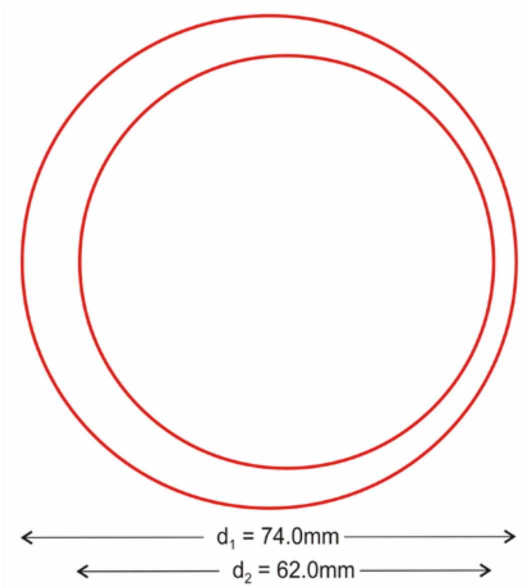
$O_3 = \pi \cdot (3.7^2cm^2 - 3.1^2cm^2) = 12.8cm^2$

The sum of these three surface parts is

$O = 257.5cm^2$

With $257.5cm^2 \cdot d \cdot 7.9gcm^{-3} = 153g$,
the thickness d of the stainless steel sheet can be calculated under the precondition that the thickness is uniform all over:

d = 0.075cm



2) *Where is the centre of mass of the entire – empty - bowl under the precondition that the thickness of the sheet is uniform all over?*
Can the centre of mass be where you calculate it under this precondition?

Centre of mass of a thin walled spherical cap (this formula can be found in the internet):

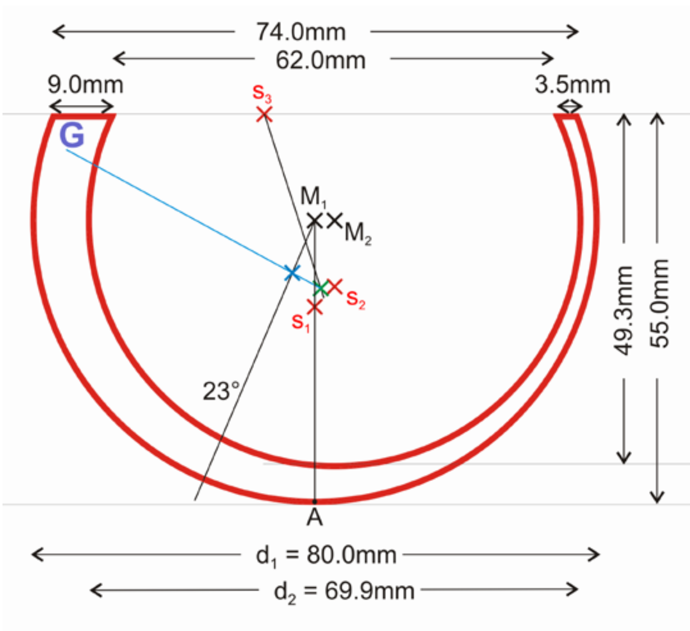
$z_s = -(R – h/2)$

e.g. for the centre of mass s_1 of the outer spherical cap

$s_1 = -(39.3-54.3/2) = -12.15$mm
means s_1 is located 12.15mm under M_1

ditto

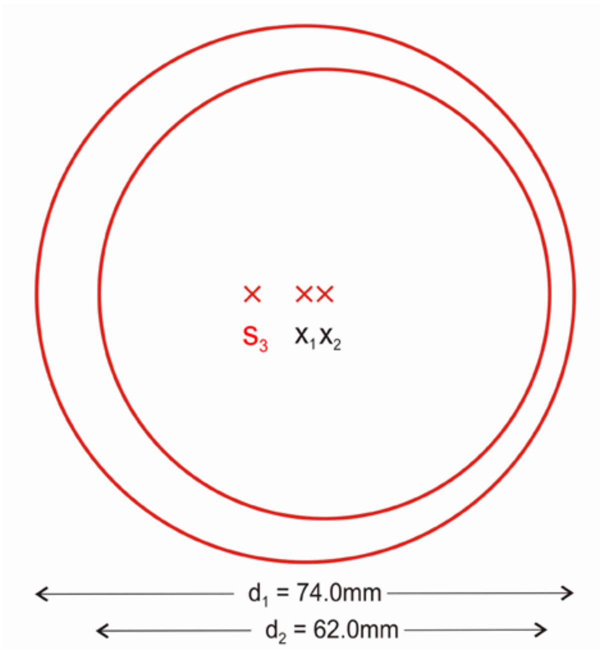
$s_2 = -(34.3-50.0/2) = -9.3$mm under M_2



The centre of mass s_3 of the eccentric circular ring can be calculated from

$s_3 = -(x_1m_1 - x_2m_2)/(m_1 - m_2)$
$= -t/((d_1/d_2)^2-1) = -3/((74/62)^2-1) = -7.1$mm

with $t = x_1 – x_2 = 3$mm, and $x_1 = 0$,
and $m_1 = \pi r_1^2 d\rho$ ditto m_2



These three centres of mass are marked in the upper drawing. From s_1 and s_2 you can calculate the common centre of mass of both of these (lever rule).
From this and s_3 you can calculate the centre of mass of the entire empty bowl (marked with a green cross).
This centre of mass is on the right side in regard to the connection line between M_1 and A. Therefore, the empty bowl should incline to the right. This is obviously not the case.

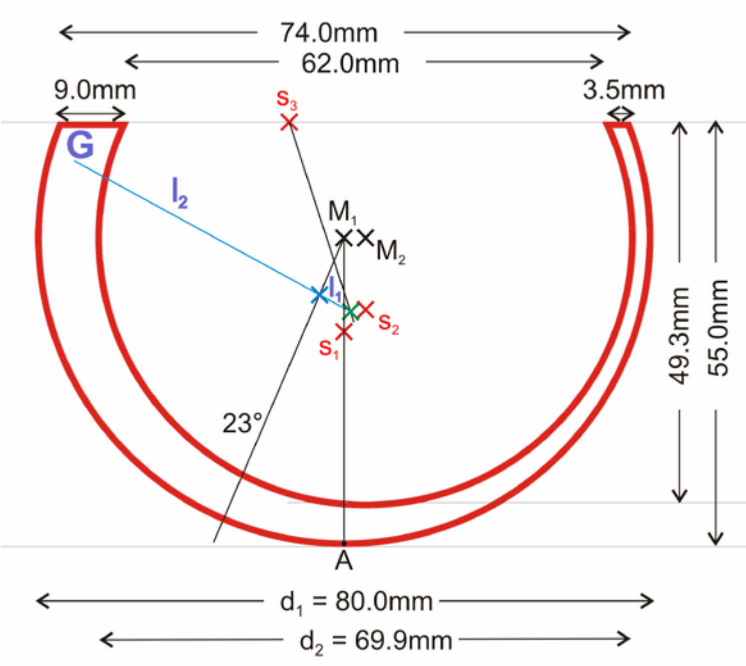
This centre of mass must be on the line at 23° from M_1. Only then would the empty bowl incline to the left under this angle.

The simplest method to correct this would be to add an additional mass G on the left.
This mass G could be hidden in the hollow part between the two spherical caps.
Other possibilities are that the thickness of the stainless steel sheet is not uniform all over or that the hollow space between the spherical caps is filled with a liquid, sand or other material. Perhaps there are even more possibilities.

3) *How can you determine the additional mass G under the border of the bowl (without destroying the bowl)?*

Assumption: the additional mass G is on the left under the thicker border of the bowl.

Since the complete mass of the bowl is fixed, the thickness of the sheet must decrease. G must be so great enough to shift the centre of mass of the complete bowl on the line from M_1 at 23° (blue cross).



On the one hand, the condition must exist that (with d' as the unknown new thickness of the sheet)

(1) $G + 257.5\text{cm}^2 \cdot 7.9\text{gcm}^{-3} \cdot d' = 153\text{g}$

On the other hand, the ratio between l_1/l_2 must conform with the ratio of the mass G to the complete new mass calculated with the new thickness d'
(lever rule; determination of centre of mass)

(2) $l_1/l_2 = G/(257.5\text{cm}^2 \cdot 7.9\text{gcm}^{-3} \cdot d')$

From the drawing, you can get the ratio $l_1/l_2 = 3/23.5 = 0.13$

From equation (1) and (2), $d' \approx 0.067\text{cm} = 0.67\text{mm}$ and $G \approx 18$ g.

The thickness of the sheet decreases a little (from 0.75mm to 0.67mm). The additional mass shifts the centre of mass into the right position.

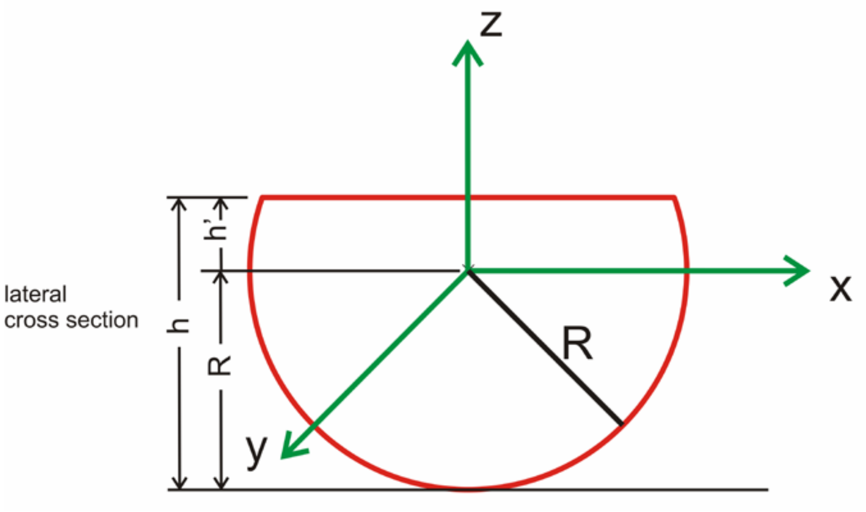
With the measurement errors already mentioned, the uncertainty of the calculation is estimated at about 10%.

4) I actually cut open a Balance bowl (see figure on right). As you can see, the assumption of an additional mass is verified. A rough estimation of these two stainless steel masses ($\rho \approx 8\text{gcm}^{-3}$) comes to about 20g
(part 1: 2.5mm x26mm x25mm, part 2: 2.5mm x14.5mm x25mm)
This conforms roughly to the calculation.
The additional masses seem to be welded firmly to the sheet. They cannot be removed without force.
The thickness of the sheet is about 0.70mm.
The bowl is probably produced in a deep-drawing process
(the company **carl mertens** does not make production details available).

5) *Calculation of the volume and the centre of mass of a spherical segment (spherical segment = massive spherical cap)*

Volume and centre of mass of a spherical segment

The formulas derived for this are not easily found explicitly on the internet or in mathematical formulary.



The z component extends from –R to h' ; the horizontal cuts have a radius dependent on z, namely $\sqrt{R^2 - z^2}$ because this deals with full circles with φ between 0 to 2π. Therefore, the spherical segment in cylindrical coordinates has the following representation:

Sphere: $-R \le z \le h', \quad 0 \le r \le \sqrt{R^2 - z^2}, \quad 0 \le \varphi \le 2\pi$

Volume segment $V_s = \int_0^{2\pi} \int_{-R}^{h'} \int_0^{\sqrt{R^2 - z^2}} r\, dr\, dz\, d\varphi$

$$V_s = \int_0^{2\pi} \int_{-R}^{h'} \frac{1}{2}(R^2 - z^2)\, dz\, d\varphi$$

$$V_s = \int_0^{2\pi} \left[\frac{1}{2}(R^2 h' - \frac{h'^3}{3} + R^3 - \frac{R^3}{3}) \right] d\varphi$$

$$V_s = \left[\frac{1}{2}(\frac{2R^3}{3} + R^2 h - \frac{h^3}{3}) \right] 2\pi$$

$$V_s = \left[\frac{2}{3} R^3 + R^2 h' - \frac{h'^3}{3} \right] \pi$$

with h' = h – R follows $V_s = \frac{h^2 \pi}{3} [3R - h]$

centre of mass $$z_S = \frac{\int z\, dV}{V_S}$$

$$z_S = \frac{\int_0^{2\pi} \int_{-R}^{h'} \int_0^{\sqrt{R^2-z^2}} r\, z\, dr\, dz\, d\varphi}{V_S}$$

$$z_S = \frac{\int_0^{2\pi} \int_{-R}^{h'} \frac{1}{2}(R^2 - z^2)\, z\; dz\, d\varphi}{V_S} = \frac{\int_0^{2\pi} \frac{1}{2}\left(\frac{R^2 z^2}{2} - \frac{z^4}{4}\right)\Big|_{-R}^{h'} d\varphi}{V_S}$$

$$z_S = \frac{\int_0^{2\pi} \frac{1}{2}\left[\frac{R^2 h'^2}{2} - \frac{h'^4}{4} - \frac{R^4}{2} + \frac{R^4}{4}\right] d\varphi}{V_S}$$

$$z_S = \frac{\frac{1}{2}\left[\frac{R^2 h'^2}{2} - \frac{h'^4}{4} - \frac{R^4}{4}\right] 2\pi}{V_S}$$

$$z_S = \frac{\left[\frac{R^2 h'^2}{2} - \frac{h'^4}{4} - \frac{R^4}{4}\right]}{\left[\frac{2}{3}R^3 + R^2 h' - \frac{h'^3}{3}\right]} = \frac{3\left(2R^2 h'^2 - h'^4 - R^4\right)}{4\left(2R^3 + 3R^2 h' - h'^3\right)}$$

Test: With h' = R, the correct calculation is thus $z_s = 0$
With h' = 0, the correct calculation is thus $z_s = -3R/8$ (this is just a hemisphere)
With h' = R/2, the correct calculation is thus $z_s = -R/8$

With h' = h – R, there is a simpler formula:

$$z_S = \frac{\left[Rh^3 - R^2 h^2 - \frac{h^4}{4}\right]}{\frac{h^2}{3}\left[3R - h\right]} = \frac{3\left[Rh - R^2 - \frac{h^2}{4}\right]}{\left[3R - h\right]}$$

The reference point is the centre of the sphere.

6) *Determination of the centre of mass of the bowl with liquid (water/wine)*
How is the centre of mass influenced by the level of the liquid?
Is the centre of mass directly above the bottom pole A?

The bowl is filled with water to a level of 44mm, i.e. about 5mm below the edge, which is quite full. Wine and water have almost the same density.

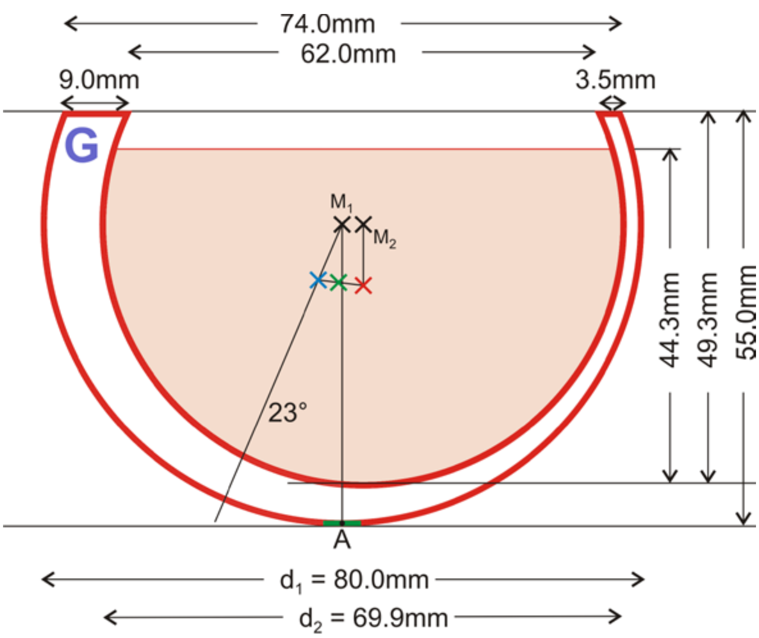
The volume of the liquid is calculated with the formula from point 5)

$V_s = 124cm^3$ (124g of water)

The centre of mass is also calculated with the formula from point 5)

$z_S = -8,1mm$

This centre of mass is perpendicular below M_2.



With the distance of the centre of mass of the bowl (blue cross) and the liquid (red cross) and the related masses, the total centre of mass (green cross) can be calculated. The distances can be evolved from the drawing.
Ideally, this total centre of mass should be located on the line between M_1 and A. This is not exactly true here, again probably due to errors in measurement.
If more liquid is filled into the bowl, the total centre of mass would shift a bit more to the right and thus result in even better stability.

The bowl stands upright even with different levels of liquid. How is this possible?

Accurate inspection of the bottom pole of the bowl shows that there is a flattening of about 0.2mm (see photograph). The black line in the photograph is a circle with the diameter of the bowl. This means that there is a small, almost flat base of about one centimetre. Therefore, small variations of the total centre of mass can be compensated.