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Hands-on physics with paper clips

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Paper clips are omnipresent. Amazing hands-on physics experiments can be done with them. Paper clips are obtainable practically everywhere and can be handled easily.

The invention of the paper clip is attributed to the Norwegian Johan Vaaler. He patented it in 1899 in Germany since Norway had no Patent Office. In 1999, a stamp appeared to conmemorate this great invention (figure 1). Vaaler did not market his invention. That happened a short time later in the United States, where a former USA-patent is even referred to. In the meantime, a variety of different forms of paper clips exists. Billions are used annually.

Fig. 1: The paper clip was invented in Norway in 1899.

Tops from paper clips

How can you make a top out of a paper-clip? Takao Sakai from Japan has described some interesting possibilities [1]. To actually build the top, turn the wire paper-clip in a circle around an axis so that the top's center of gravity is in the axis (figure 2). For this, the angle *ß* between the spokes must be about *53°*. Calculation of this angle is a nice task for first year physics students (see info-box 1). The construction, observation and understanding of the top itself is an entertaining exercise for children and scientists. For the real construction, it is better to use paper clips with a soft wire. They can even be distorted without tongs. It is not important to realize the exact angle and the circle-form. It is only essential that the center of gravity lie in the axis. Many additional top forms can be realized [2].

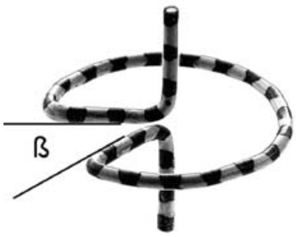

Fig. 2 : A top bent from a paper clip

The construction of a tippe-top from a paper clip is unusually simple (figure 3). Kamishina has published such a construction [3]. As you can see from the figure, the center of gravity of the top doesn't coincide with the center of the major circle. That is characteristic for the construction of the classical tippe-top, that normally consists of a part of a sphere with a stem. Starting the spin of this tippe-top is somewhat arduous since it must be touched at the circle-edge outside and therefore does not achieve high revolution speed. The effect however, is clearly visible. It works somewhat better if you push the opposite sides of the top with your forefingers (each side into the opposite direction). But that needs practice.

Fig. 3 : A tippe-top made from one big paper clip

Unfortunately, there is no simple explanation for the behavior of the tippe-top which can be formulated in a few sentences. For almost a hundred years, physicists have tried to provide an applicable description in various publications. Only few descriptive aspects can be derived from the complex mathematical expressions. As an exercise for advanced physics students, the tippe-top is discussed by Kuypers [4].

Chains with paper clips

Which shape does the curve of a chain or a flexible cable or rope suspended from its two ends take? Galileo Galilei himself asked himself this question– and answered it incorrectly. He thougth it was a parabola. The correct curve wasn't derived until the end of the 17th century, by the brothers Jacob and Johann Bernoulli as well as by Gottfried Wilhelm Leibniz. It is the so-called catenary curve, which is the cosine hyperbolicus function (cosh), that can also be expressed as sum of two exponential functions. The derivation of the catenary curve can be found in many mathematical textbooks as well as books about mechanics and thus is not shown here.

Fig. 4 : Catenary curve with 16 paper clips

The catenary curve can be constructed well with an adequate number of paper clips. The more paper clips used, the better the approximation to the ideal curve. With fewer paper clips, the length and the connection between the paper clips play a role. In figure 4 the ideal catenary curve and a parabola of the same length are sketched over a chain with 16 paper clips. You can clearly recognize that the red catenary curve corresponds with the paper clip-chain, but the blue parabola doesn't. The difference between a catenary curve and a parabola is especially distinctive with a relatively strong sag as in figure 4.

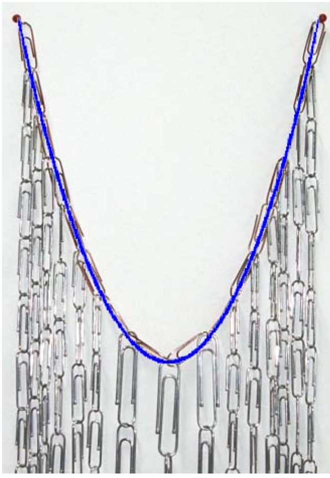

Fig. 5 : Parabola with 16 paper clips and weights. An ideal parabola (blue) is sketched over it. The entire length of the hanging paper clip weights isn't shown here.

If heavy weights hang on each link of a chain, as for example with suspension bridges, the curve really changes from a catenary curve to a parabola (see info box 2). This case can also be constructed with paper clips. In figure 5 the same chain of paper clips as in figure 4 is weighted with heavy pieces built from long chains of paper clips with a total mass at least twenty times larger than the mass of one paper clip (~ 0,37g). The number of the paper clips (the size of the mass) must have a certain relationship to the slope of the relevant section of the parabola curve: the closer to the ends of the chain, the smaller the weight (for example of the section of roadway on a bridge) which the link carries.

In this case the agreement with the parabola (blue) can be recognized clearly. By the relatively heavy weight suspended at the lowest link of the chain, the curve is pulled further downward than the catenary curve.

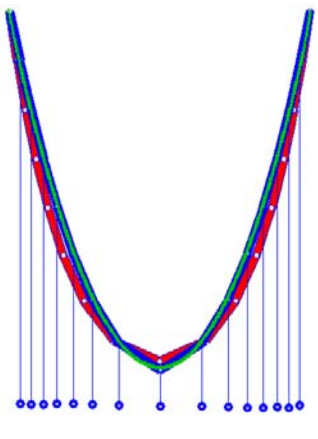

Fig. 6 : Simulation of a catenary curve and a parabola using 'Interactive Physics'

With physics simulation programs like Interactive Physics [5] or XYZet [6] you can also illustrate the situations described very nicely. In figure 6, you can see the simulation of a catenary curve with 16 unweighted links (thick, red line), a 'suspension bridge' equipped with corresponding weights, (thick, blue parabola), and an ideal parabola (thin, green line), all overlaid on one another. With a real suspension bridge, the vertical suspender cables are equidistant from one another (this can also be simulated with the program). In figure 6, however, the points on the main suspension cable where the vertical suspender cables are fastened are equidistant from one another.

In the WEB you can find many links under the term ‘catenary curve’, and also historical remarks and derivations. Furthermore, there are very descriptive applets that clarify the difference between the catenary curve and the parabola.

Info-box 1 (the Sakai top)

In figure 7 the top is pictured from above. If the angle between the spokes is too large or too small, the total center of gravity will not be located in the middle of the circle. For calculation of the correct angle you can limit yourself to the calculation of the center of gravity of both spokes and the opposite small arc with the length s. The other (red) parts of the arc are symmetrical to the center of the circle and therefore don’t need to be taken into consideration. It is helpful to introduce the half spoke angle α.

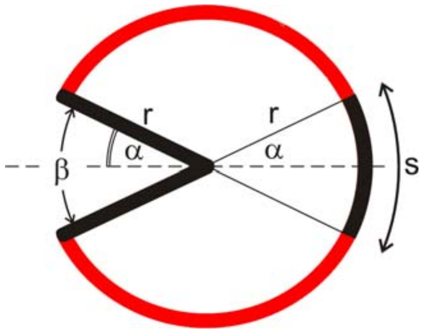

Fig. 7: A view of the paperclip top from above

In figure 8 the spokes and arc are shown by heavy lines. The center of the circle becomes the origin of the coordinate system. The distance between the center of gravity of the arc and the origin is x_1. Because of the symmetry to the x-axis, if s is the length of the arc, the center of gravity can be calculated with the line integral

$$x_1 = \frac{\int xds}{s} = \frac{1}{s}\int_{-\alpha}^{\alpha} r \cdot \cos\varphi \cdot r \cdot d\varphi = \frac{2 \cdot r^2}{s}\sin\alpha$$

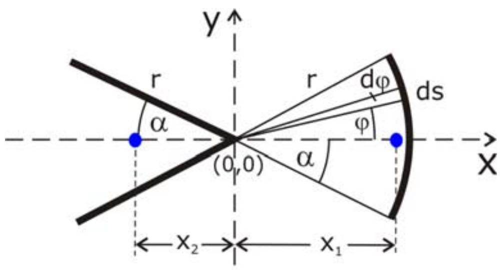

Fig. 8: For calculation of the center of gravity only the spokes and the arc opposite these are taken into consideration.

If ρ is the density of the wire per unit length and A the cross sectional area, then the mass of the arc is $m_1 = s \cdot \rho \cdot A$. With respect to the origin the arc has a torque

$M_1 = m_1 \cdot x_1 = 2 \cdot r^2 \cdot \sin\alpha \cdot \rho \cdot A$.

The center of gravity of the spokes is at $x_2 = r/2 \cdot \cos\alpha$, the mass is $m_2 = 2 \cdot r \cdot \rho \cdot A$. The torque generated by the spokes with respect to the origin is

$M_2 = m_2 \cdot x_2 = r^2 \cos\alpha \cdot \rho \cdot A$.

When the expressions for the two torques are equated, the result is $\tan\alpha = 0.5$, i.e. $\alpha = 26.565^\circ$. The angle between the spokes is then $ß = 2\alpha = 53.13^\circ$.

Info-box 2 (suspension bridge parabola)

Strongly simplified and idealized, the form of the curve of the main cable of a suspension bridge can be derived in the following way:
Three forces, whose vectorial addition must compensate itself exactly (figure 9), act in a point P of the main cable of a suspension bridge. First, the force G of a part of the street with the length x acts vertically downward. Secondly, a horizontal force S is exerted by the tension of the cable. This force is constant over the whole cable. Thirdly, a force F acts in the direction of the tangent to the cable. This tangential force corresponds exactly to the slope at point P.

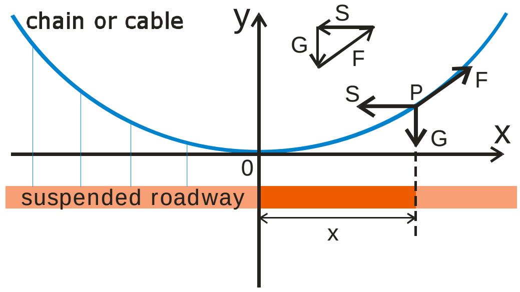

Fig. 9: A parabola emerges as a curve form for the main cable of a suspension bridge.

Let's take µ as the weight per unit of length of the roadway suspended at the cable. The coordinate origin 0 is located at the root-point of the curve. The weight G acting at the point P is then exactly G = µ·x. If we mark the height of the cable with y at the point x, then the slope at this point is

$$y' = \frac{G}{S} = \frac{\mu \cdot x}{S}$$

Through integration from this equation we get

$$y = \int \frac{\mu \cdot x}{S} dx = \frac{\mu}{2S} x^2 + C$$

Since the origin 0 of the curve is located at the root-point, the integration constant is C = 0 . Consequently, the form of the curve is a parabola.

References:

[1] **Sakai, Takao:** Topics on tops which enable anyone to enjoy himself, 数理科学 (Mathematical Sciences), No. 271, January 1986, pp. 18-26 (in Japanese)
[2] **Ucke, Christian**, Professor Sakai's Paper-Clip Tops**,** Physics Education (India) **19** No. 2, July-September 2002, pp. 97-100
[3] **Kamishina, Yoshio**: Proceedings of International Workshop on Hands-On Activities for In-School and Out-of-School Learners Focusing on the Marginalized Youth, Pattaya, Thailand 1999
[4] **Kuypers Friedrich**: Classical Mechanics (in German), Wiley-VCH-Verlag, Weinheim
[5] http://www.interactivephysics.com/ (in English)
[6] http://www.ipn.uni-kiel.de/persons/michael/xyzet/ (in English)

Additional material can be downloaded from:
Video of a paper clip tippe-top: http://www.wiley-vch.de/berlin/journals/phiuz/05-01/stehaufkreisel.mpg
Catenary curves with simulation programs: http://www.wiley-vch.de/berlin/journals/phiuz/05-01/kettenlinie.zip

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