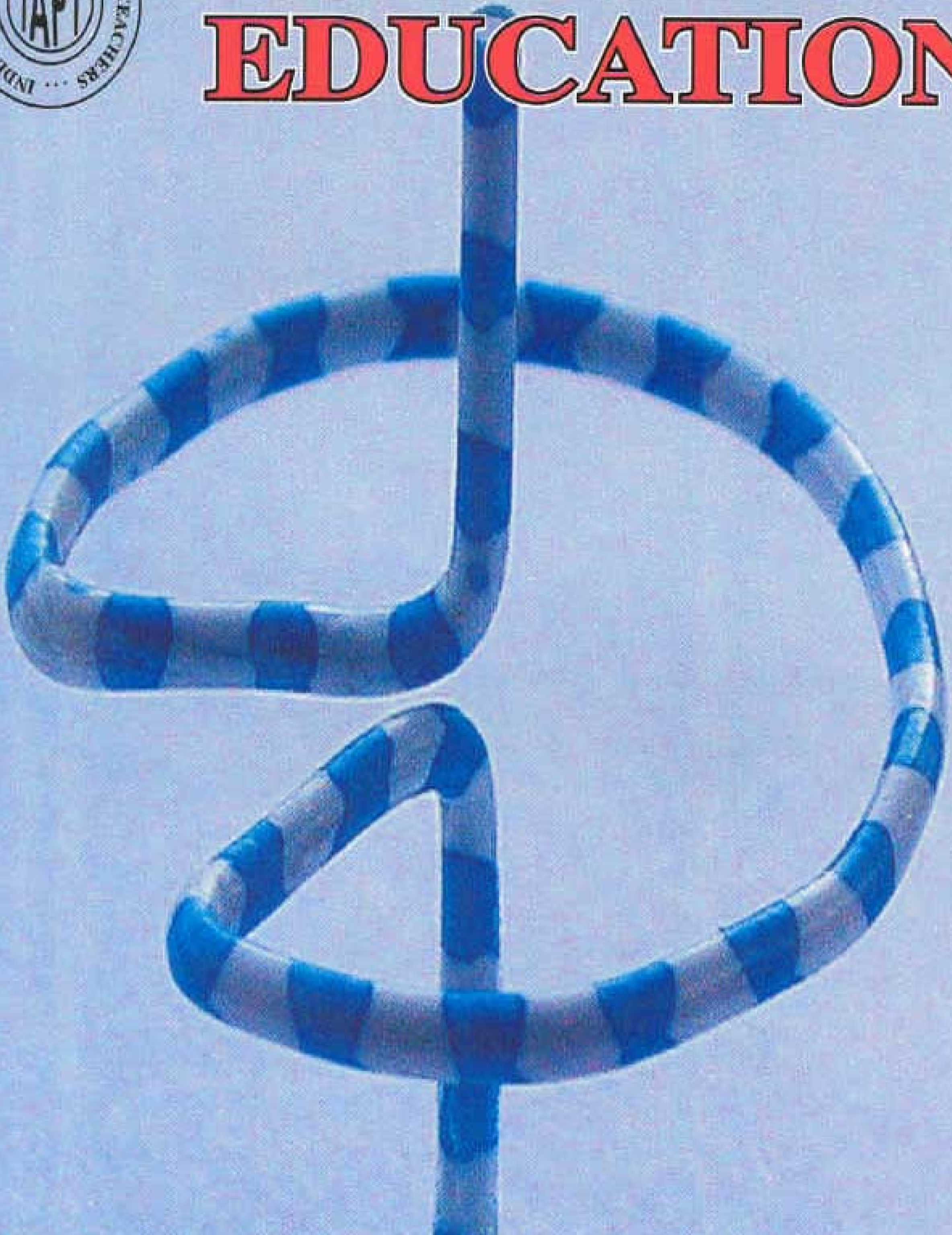
VOL. 19 NO. 2 JULY-SEPTEMBER 2002 ISSN 0970-5953

PHYSICS EDUCATION



Paper Clip Top

Professor Sakai's Paper-Clip Tops

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How can you make a top out of a paper-clip? Takao Sakai from Japan has described some interesting possibilities.[1] Since Japanese publications are difficult to obtain and hard to read, I have asked a Japanese colleague to translate the most important parts for me. Some important parts of the physics are even clear without knowledge of the Japanese language. I will present Takao Sakai's ideas combined with some of my own reflections (Figures 4, 5 and following sections). To actually build the top turn the wire paper-clip in a circle around an axis so that the top's centre of gravity is in the axis (Figure 1). For this, the angle β between the spokes must be exactly 53.13°. Calculation of this angle is a nice task for first year physics students. The construction of the top itself is an entertaining exercise for children and scientists.

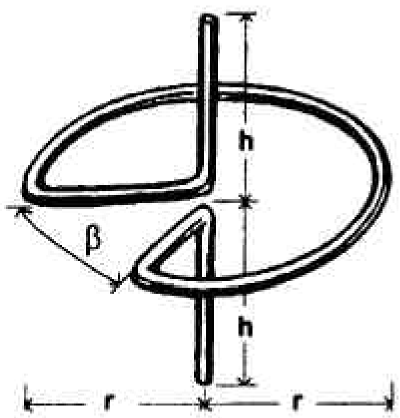

Figure 1. A paper-clip can be bent into shape to form a top.

The top is pictured from above Figure 2. If the angle between the spokes is too large or too small, the total centre of gravity will not be located in the middle of the circle. For calculation of the correct angle you can limit yourself to the calculation of the centre of gravity of both spokes and the opposite small arc with the length s. The other parts of the arc are symmetrical to the centre of the circle and therefore don't need to be taken into consideration. It is helpful to introduce the half spoke angle α.

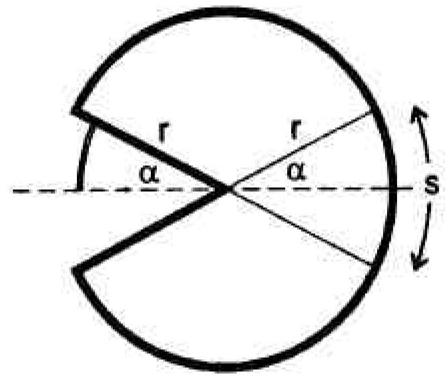

Figure 2. A view of the paper-clip top from above.

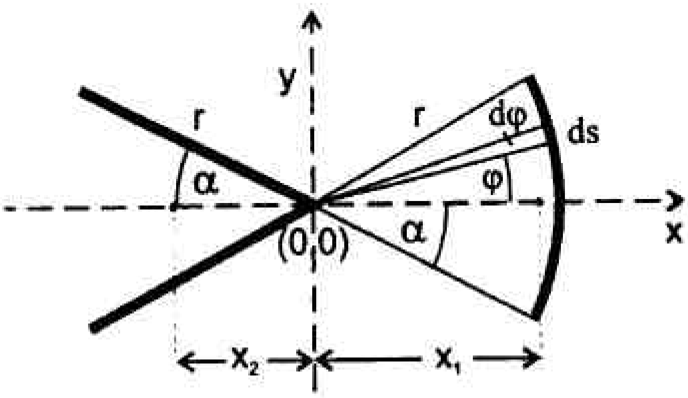

Figure 3. For calculation of the centre of gravity only the spokes and the arc opposite these are taken into consideration.

Physics Education is a quarterly journal published by the Indian Association of Physics Teachers in Association with the University of Pune, Indian Space Research Organization & Department of Atomic Energy

In Figure 3 the spokes and arc are shown by heavy lines. The centre of the circle becomes the origin of the coordinate system. The distance between the centre of gravity of the arc and the origin is X_1. Because of the symmetry to the x-axis, if s is the length of the arc, the centre of gravity can be calculated with the line integral

$$x_1 = \frac{\int x ds}{s} = \frac{1}{s}\int_{-\alpha}^{\alpha} r.\cos\varphi . r. d\varphi = \frac{2.r^2}{s}\sin\alpha$$

If ρ is the density of the wire per unit length, then the mass of the arc $m_1 = s.\rho$. With respect to the origin the arc has a torque $M_1 = m_1 X_1 = 2.r^2.\rho\sin\alpha$ The centre of gravity of the spokes is at $X_2 = r/2.\cos\alpha$, the mass is $m_2 = 2.r.\rho$. The torque generated by the spokes with respect to the origin is $M_2 = m_2 x_2 = r^2.\rho\cos\alpha$.

When the expressions for the two torques are equated the result is $\tan\alpha = 0.5$, i.e. $\alpha = 26.565°$. The angle between the spokes is then $\beta = 2\alpha = 53.13°$.

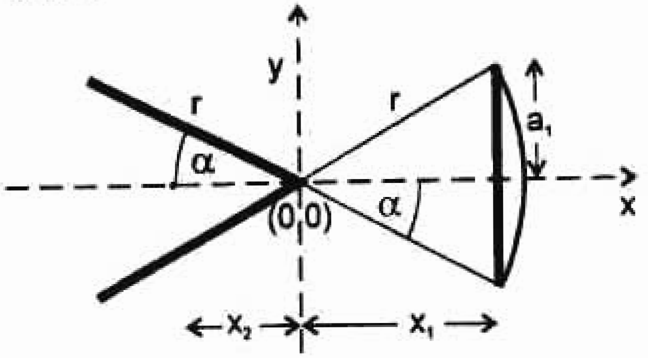

Figure 4. The arc is replaced by a chord for approximate calculation of the angle.

The previous observations require intergrations which cannot be assumed to be usual in schools. However, the angle α can be calculated approximately without integral calculus. To do this, the arc can be replaced by the chord (Figure 4). The centre of gravity of this chord is at $x_1 = r\cos\alpha$, the mass is $m_1 = 2.a_1\rho = 2r\rho\sin\alpha$. Thus the resulting torque $M_1 = m_1 X_1 = 2.r^2\rho\sin\alpha\cos\alpha$. If you equate this with this M_2, which has already been determined, the result is $\sin\alpha = 0.5$ resp. $\alpha = 30°$. This angle is somewhat larger than the exactly claculated angle of 23.65° since the centre of gravity of the chord is nearer to the centre of the circle.

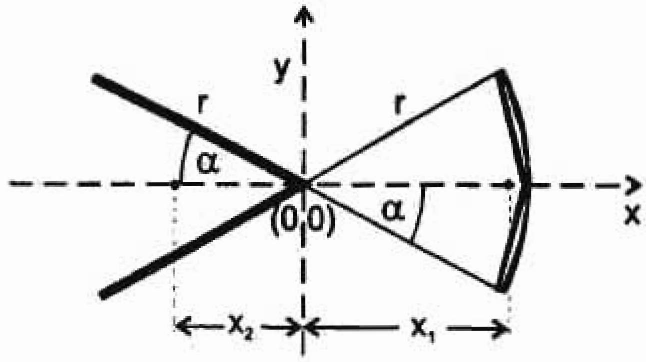

Figure 5. An approximation in which the arc is replaced by two chords is even better.

An approximation in which the arc is replaced by two chords is much better (Figure 5). Although a transcendental equation without an explicit solution results after an elementary calculation for the angle α, this can be solved quickly with modern computers. The angle here is 27.2°, and that is very close to the exact angle.

The top can be bent into shape with a small pair of pliers. You can even use your fingers to bend the paper-clip, but this won't work out as well. Paper-clips with rounded ends are particularly suitable. For the top part of the axis to equal the bottom part of the axis and be just as big as the radius, there is a small calculation to be considered since the upper part of the axis must be about 1/10 of the entire length of the paper-clip. Bending the arc and getting the correct angle between the spokes is not very easy. However, this is not very critical since you can almost always still place the entire centre of gravity onto the circle's axis through additional bending. It is a nice task to have each student in a class make a top and then see which one functions best.

The top can be spun with your fingers and reach several thousand rotations per mintue. When it spins very fast, you see only the arc and the axis. You have the impression of a circular ring floating freely in the air.

When the top has been spun by hand, the number of rotations per minute can be estimated in the following way. If you hold the top's axis between your index finger and thumb, your fingers spin it off at a speed v of about 0.1 m/s. The diameter of the wire of a small paper-clip is about 1 mm, the radius r of the top's axis thus 0.5 mm. From this $\omega = v/r = 0.1\ \mathrm{ms^{-1}}/0.5\ \mathrm{mm} = 200\ \mathrm{s^{-1}}$ respectively $f = 200\mathrm{s^{-1}}/2\pi = 32\mathrm{s^{-1}} = 1900$ rpm. This estimation should only be understood as a rough approximation.

If the top's centre of gravity isn't located in the top's axis, the top is not balanced. As with an unbalanced automobile tyre, strong forces act upon the wheel bearing, but the axle remains tight. With the car tyre, small weights are fastened onto the rim as a counterbalance. With the top, the axis makes irregular movemenets. With the paper-clip top, you can usually manage to get the centre of gravity onto the top's axis by bending the wire.

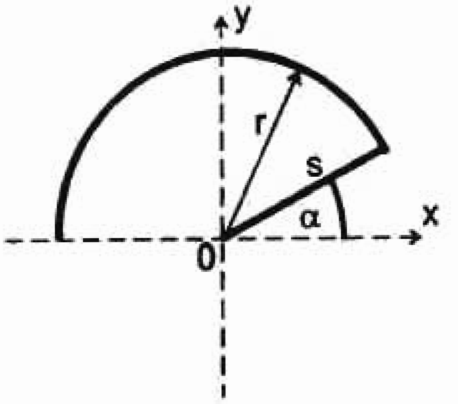

Figure 6. Calculation of the inertia moments. Here is shown only that half of the top in the *x-y*-plane.

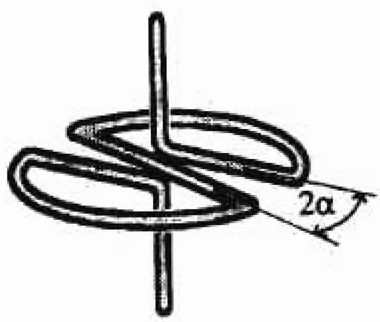

Figure 7. Construction of a symmetrical Sakai top.

The Sakai top is a so-called non-symmetrical top in the sense of theoretical physics, i.e. the top's moments of inertia in regard to two perpendicular axes at the level of the arc are not the same ($I_x \neq I_y$). They can be calculated with somewat more exertion. From Figure 6 one can deduce.

$$I_x = 2\left[\int_0^r (s\sin\alpha)^2 \rho\, ds + \int_\alpha^\pi (r\sin\varphi)^2 \rho r\, d\varphi + \frac{1}{3}\rho h^3\right]$$

$$= 2\rho r^3\left[\frac{\sin^2\alpha}{3} + \frac{\pi}{2} - \frac{\alpha}{2} + \frac{\sin 2\alpha}{4} + \frac{1}{3}\left(\frac{h}{r}\right)^3\right]$$

$$I_y = 2\left[\int_0^r (s\cos\alpha)^2 \rho\, ds + \int_\alpha^\pi (r\cos\varphi)^2 \rho r\, d\varphi + \frac{1}{3}\rho h^3\right]$$

$$= 2\rho r^3\left[\frac{\cos^2\alpha}{3} + \frac{\pi}{2} - \frac{\alpha}{2} - \frac{\sin 2\alpha}{4} + \frac{1}{3}\left(\frac{h}{r}\right)^3\right]$$

The inertia moment with respect to the z-axis is

$$I_z = 2\rho r^3\left(\frac{1}{3} + \pi - \alpha\right)$$

With a certain relationship of h/r ($1.62 \leq h/r \leq 1.68$), I_z is just the middle moment of interia between I_x and I_y. Free rotation of the top around the middle moment of inertia's axis is unstable. If a top with such a configuration is spun on a hard surface, this can mean that it won't rotate smoothly any more below a certain rotation speed.

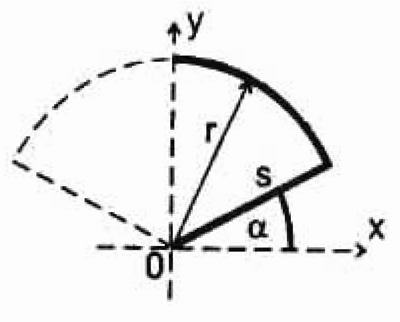

Figure 8. View of a part of the top from Figure 7.

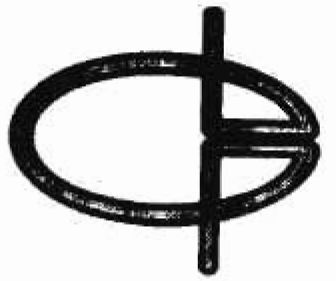

Figure 9. Another Sakai top.

In Figure 7 the possibility of building a symmetrical paper-clip-top according to Takao Sakai is shown. If the angle α between the spokes has the right value, then $I_x = I_y$. To calculate that angle look at Figure 8. In a way very similar to the former calculation of the inertia moments the result for the fourth part of I_x and I_y is

$$I_x = \int_0^r (s \sin\alpha)^2 \rho ds + \int_\alpha^{\pi/2} (r \sin\varphi)^2 \rho . r . d\varphi$$

$$= \rho r^3 \left[\frac{\sin^2\alpha}{3} + \frac{\pi}{4} - \frac{\alpha}{2} + \frac{\sin 2\alpha}{4} \right]$$

$$I_y = \int_0^r (s \cos\alpha)^2 \rho ds + \int_\alpha^{\pi/2} (r \cos\varphi)^2 \rho . r . d\varphi$$

$$= \rho r^3 \left[\frac{\cos^2\alpha}{3} + \frac{\pi}{4} - \frac{\alpha}{2} - \frac{\sin 2\alpha}{4} \right]$$

Equating $I_x = I_y$ yields $\tan 2\alpha = 2/3$ resp. $2\alpha = 33.69°$.

Questions concerning variations of the Sakai tops present themselves. What effect do the small rounded edges have, with which the tops actually must be constructed? How does the angle between the spokes change when the spokes run symmetrically upwards or downwards to the top's axis, with the centre of gravity still remaining in the top's axis and the level of the circle's ring? Where must the top's axis be if both spokes run parallel (Figure 9) and the centre of gravity of the circle's ring and spokes should still be in the top's axis? Paper-clip tops with a square section instead of the circular part can also be constructed in a similar way and can be calculated without integrals.

References

1. Takao Sakai (in Japanese), "Topics on tops which enable anyone to enjoy himself", *Mathematical Sciences*, No.271, Jan. 1986, pp.18-26.
2. Christian Ucke, (in German), Kreisel aus Büroklammern Physikalische Blätter 54 (1998), Heft 5, 440-442.
3. An animated gif-picture of a Sakai-Top can be seen in the internet under: http://www.e20.physik: tu-muenchen.de/~cucke/ftp/lectures/sakaigir.htm.