JUMPING TOYS:

A TOPIC FOR INTERPLAY BETWEEN EXPERIMENT AND THEORY

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Content:

5' **1) Standing high jump demonstrated by the flea and man; presentation of the toy**

20' **2) Simple experiments and theory**

10' **3) Higher level experiments and theory; modelling**

S 35'

Ladies and Gentleman, dear colleagues,

I am going to talk about toys, especially about jumping toys – and animals. Probably all of you know this small toy. But to be sure, I will give everybody such a toy now. This is normally a dangerous idea during a talk because the members of the audience can play during the talk. But my idea is to give you information not only through hearing and seeing but also through feeling.

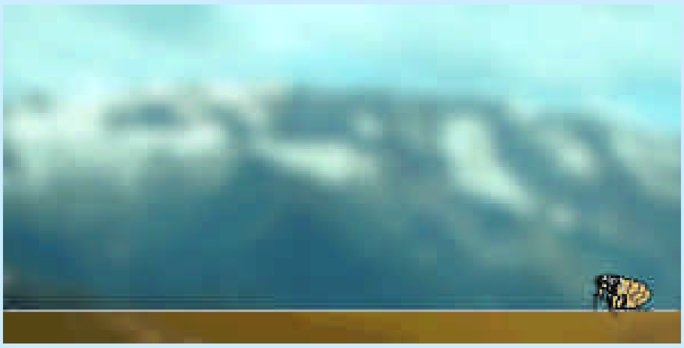
The English trade name for the toy is ‚Springy Smiley Face‘.

Everybody can take a sample or even two. There are about 200 toys.

Furthermore, I would like to change the subtitle from 'interplay between theory and experiment' to 'interplay between experiment and theory'. The experiment is for me the first and most important operation.

Here is the content of my talk. I hope to need not more than 35 minutes.

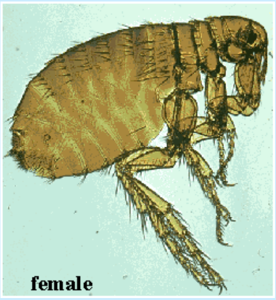
jumping animals



Probably all of you are familiar with this small animal, whether through your own experience or not.

This animated picture comes from the Russian Zoological Institute in St. Petersburg, which explores all properties of the fleas (http://www.zin.ru/Animalia/Siphonaptera/index.htm)

jumping animals



$\approx$ 2mm

Pulex irritans
(= human flea)

height of the jump h $\approx$ 0.5m

acceleration distance d $\approx$ 2mm

acceleration a = h·g/d = 2500ms^{-2}
$\approx$ 250g

(uniform acceleration assumed; g = 10ms^{-2} = acceleration of gravity)

man's acceleration $\leq$ 3g

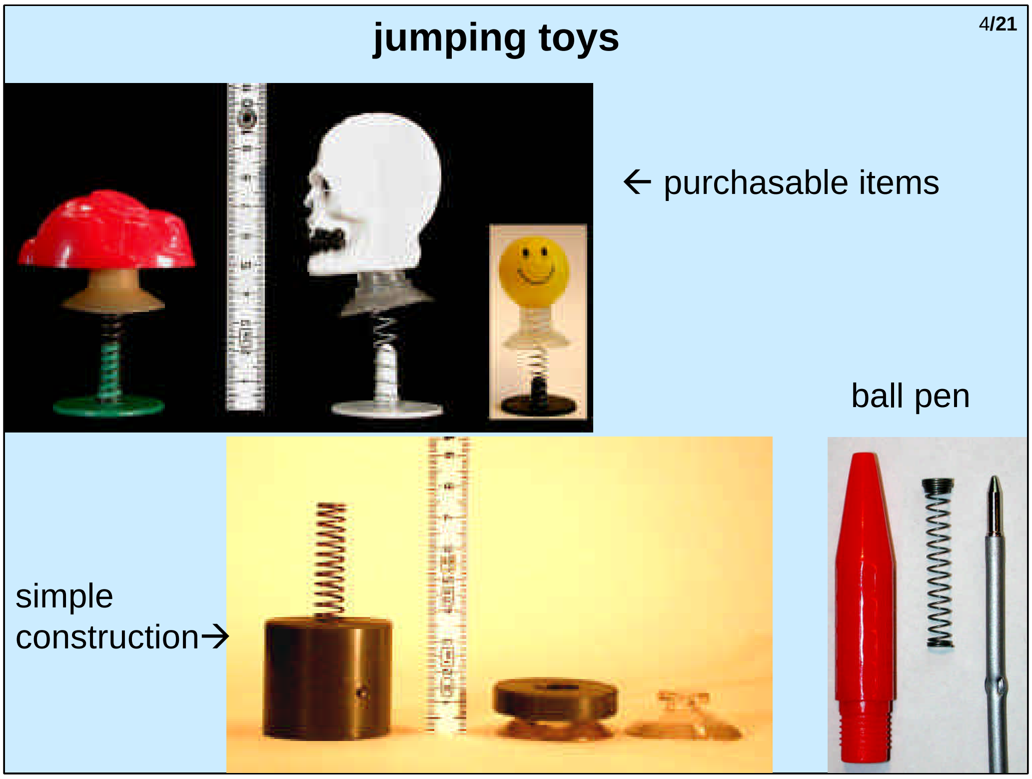
From $v=\sqrt{2gh}$ *and* $v=\sqrt{2ad}$
results $a=hg/d$

A flea jumps up to a height of about $h = 0.5$ *m*. It accelerates across a distance of about $d = 2$ *mm* it. This leads to an acceleration of $a = h{\cdot}g/d = 0{,}5\ m{\cdot}g/0.002\ m = 2500ms^{-2} = 250g$ (g = 10ms^{-2} = acceleration of gravity; uniform acceleration assumed). Since the jumping height of a flea is strongly influenced by air resistance and the acceleration is not uniform, it has, in reality, a greater initial acceleration. There are other animals with an even greater acceleration. Biologists have investigated very accurately the jumping mechanism of the flea. This is a topic which I cannot explain here.

The jumps of fleas and other animals are difficult to measure and not very reproducible.

This is a good reason for investigating a toy which gives better reproducible results.

A man can only achieve up to 3g with a standing high jump.



A small toy known as a jumping animal or pop-up makes some investigations easier and can illuminate the physics of jumping. The toy itself consists of a base, a spring, a suction cup and a head. You have to press the cup onto the base and thus load the spring. After some time the cup will loosen itself and the toy will jump up.

Here you can see several different shapes of the toy. I started my own investigations with the toy on the left. It was available in Germany one year ago but is out of production just now but will be available in a few months again. At the ‚Oktoberfest‘ in Munich I once got this strange item. The one on the right side is what you have in your hands.

You can build this toy easily by yourself. You need a compression spring with a length of about 6cm and a spring constant of about 500 Nm^{-1}. In shops for household goods you can obtain simple suction cups with a hook, as used in bathrooms or kitchens. The hook must be removed.

The difference to the purchasable toy is that the base stays on the floor after the jump.

The use of appropiate ball pens is even simpler and gives almost the same properties. Probably many of you remember experiments like this with ball pens in the school..

jumping toys

Questions:

Which toy will achieve the greatest height?

Which toy will start first?

What will happen if you remove the head?

And what if you detach the spring from the base?

Which height will the toy achieve jumping upside down?

How much force do you need to compress the spring?

Which weight must be attached on to the head of the toy so that it will not jump at all?

How much time will the starting process of the toy need?

………………………………..

If children have several of these toys then a lot of questions arise:

Which toy will achieve the greatest height? Which toy will start first?

Children will start soon to discover more properties:

What will happen if you remove the head? And what, if you detach the base from the spring? Which height will the toy achieve jumping upside down?

And you can ask them questions: How much force do you need to compress the spring? The head will undergo an acceleration: When will the acceleration be at its maximum? How much time will the starting process of the toy need? Which weight must be attached on the head of the toy so that it will not jump at all? What will happen if you attach the base to the ground?

And more questions ………

jumping toy

Simple experiments and calculations:

Measured height of the jump $h = 1.2m$ (±10%)

=> $E_{pot} = mgh = 0.0145kg \cdot 10ms^{-2} \cdot 1.2m = 0.17J$

Compressing on a – kitchen - scale

$F \approx 19N$ (= 1.9kg; ±10%); $d \approx 3.2cm$ (±10%)

Spring stiffness $c = F/d \approx 590Nm^{-1}$

=> $E_{spring} = 0.5 \cdot c \cdot d^2 = 0.5 \cdot 590Nm^{-1} \cdot 0.032^2m^2 = 0.30J$

((=> $h = E_{springr}/mg = 2.1m$))

I will now talk about experiments with this toy and not with the one you have because I made all experiments with this toy.

You can do these experiments on your own with your toy with other parameters.

The first simple experiment is to measure the jumping height and calculate the potential energy.

Another simple experiment is to compress the toy on to a – kitchen - scale and to measure the weight and the compression distance.

From that you can calculate the spring stiffness and the energy stored in the spring.

As you can see, there is a great difference, which I will explain later.

The spring stiffness of the toy you have in your hands is about $400Nm^{-1}$.

The spring stiffness of the ball pen springs is about 200 to 300 Nm^{-1}.

These values are comparable to that one I have here.

jumping toy

Initial acceleration of the head

$$a = F/m_1^* - g \approx 1900\text{m/s}^2 = 190\text{g}$$

(m_1^* = head + suction cup + 1/3 spring = 0.00984kg;

g = 10m/s^2 = acceleration of gravity)

the mass of the spring cannot be disregarded

Initial acceleration of the base $a = F/m_3^* - g \approx 525\text{g}$!!!

(m_3^* = base + 1/3 spring = 0.00369kg)

The asterisk always means that the mass of the spring has to be considered, but I am going to omit it in future to avoid overloading the formulas

On a very simple level it is possible to calculate with Newton's second law the initial acceleration of the head and, as you can see, it is remarkably high.

When you let the toy jump upside-down the acceleration of the base is even incredibly high.

jumping toy/Springy Smiley Face

Simple experiments and calculations:

Measured height of the jump $h = 0.3m$ (±20%)

Compressing on a – kitchen - scale

$F \approx 7.8N$ (= 0.78kg; ±10%); $d \approx 1.5cm$ (±10%)

Spring stiffness $c = F/d \approx 520Nm^{-1}$

Mass Head $m_{Head} = 4{,}510g$
Mass Spring $m_{Spring} = 0{,}391g$
Mass Base $m_{Base} = 1{,}174g$

Initial acceleration of the head

$a = F/m_1^* - g \approx 1700m/s^2 = 170g$

(m_1^* = head + suction cup + 1/3 spring = 0.00464kg)

I add some experiments with that toy you have have in your hands.

The first simple experiment is to measure the jumping height.

Another simple experiment is to compress the toy on to a – kitchen - scale and to measure the weight and the compression distance.

From that you can calculate the spring stiffness.

The spring stiffness of the ball pen springs is about 200 to 300 Nm^{-1}.

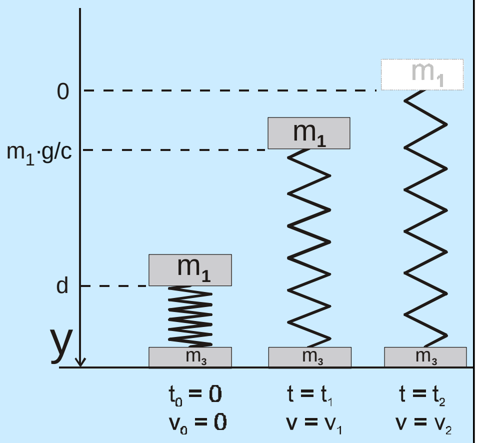
These values are comparable to that one I have here.

jumping toy

Maximum velocity of the head

$$\frac{c}{2}d^2 - \frac{c}{2}\left(\frac{m_1 g}{c}\right)^2 = m_1 g\left(d - \frac{m_1 g}{c}\right) + \frac{m_1}{2}v_1^2$$

spring energy *potential energy* *kinetic energy*



$$v_1 = -\sqrt{\frac{c}{m_1}}\left(d - \frac{m_1 g}{c}\right) \approx -8ms^{-1} \approx -30kmh^{-1}$$

The spring is compressed, and at the time $t_0 = 0$ the head starts with maximum acceleration. The time t_1 is when the mass m_1 (head + suction cup) achieves the position where the head is in the equilibrium situation. Equilibrium means the situation when the spring is not compressed and the head is in equilibrium with the spring. The time t_2 characterizes the position of the end of the spring without m_1; it is the equilibrium position of the spring without m_1.

The head will attain its maximum velocity v_1 at the time t_1. Conservation of energy leads to the following equation.

If you calculate the velocity the result is $v_1 = 8ms^{-1}$ or about 30 km/h. This is not a high velocity when you compare it with the high value of the acceleration.

The result is negative because the direction of the axis points downwards.

jumping toy

Calculated height of the jump
(conservation of linear momentum)

$$m_1 v_1 = (m_3 + m_1)v_3$$

v_3 = mean velocity of the whole toy after leaving the floor

$$h_3 = \frac{v_3^2}{2g} = \frac{c \cdot m_1}{2g(m_3 + m_1)^2}\left[d - \frac{m_1 g}{c}\right]^2 \approx 1.4m$$

To calculate the height of the jump you need the principle of conservation of momentum. t_3 should be when the bottom mass m_3 leaves the floor.

The head of the toy must pull the base (and partly the spring) when leaving the floor.

Thus you get the height. The result is realistic when compared with the measured height.

A common mistake here is to use the entire spring energy for calculating the height. This is not allowed because there are energy losses due to friction, and some energy is stored in the oscillation of the toy, as you will see later.

The height as a function of the head's mass m_1 is a complicated, nonlinear function which has a maximum. It is interesting to see that the design of the toy is almost optimized with regard to the jumping height.

jumping toy

Estimation of the time from start until achieving maximum velocity (≈ leaving the floor)

$$t = \sqrt{\frac{2s}{a}} \approx 0.0082s = 8.2ms$$

(s = 0.032m; a = 95g; this is half of the initial acceleration; uniform acceleration assumed)

You can also roughly estimate the starting time.

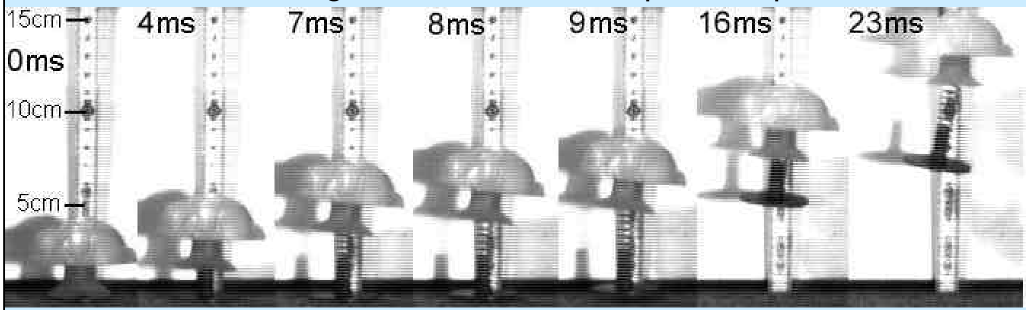
This formula is only valid for uniform acceleration!

You need calculus to calculate the time exactly.

8 milliseconds is too fast to take a normal video of this process. The time between two (half) video pictures is 20ms (PAL system).

jumping toy

frames of a digital video with 1000 pictures per second



Video

2000 pictures/second
146 pictures = 73 ms

With the help of colleagues I made videos of the jump with a high-speed digital video camera, with 1000 and 2000 pictures per second.

The figure above shows some pictures taken from the video.

At 0ms the toy starts; at 7ms the head of the toy reaches its maximum velocity and the base leaves the floor; at 9ms the spring is stretched to its maximum; at 16ms the spring is minimally stretched; at 23ms the spring is again maximally stretched. The pictures are not sharp because the higher the speed of the digital video camera, the worse the resolution.

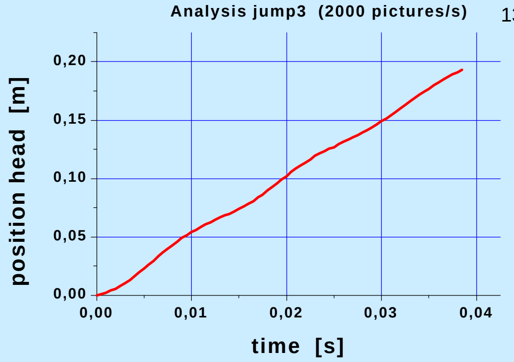
The spring oscillates. This can not be observed with the naked eye because the oscillating frequency is about 70Hz.

The high speed video camera costs about 10000 US-Dollar. This is too expensive for a normal school.

jumping toy

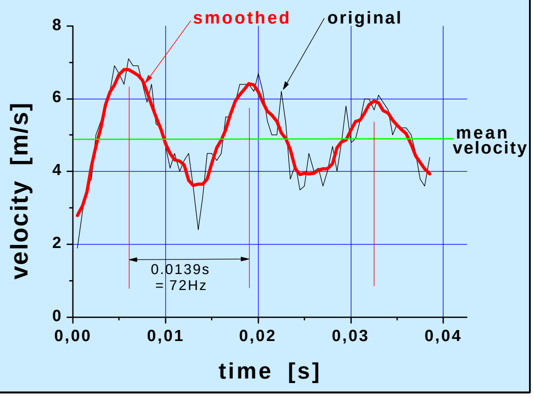
Position

Experimental results using video data analysis programs (DIVA, Coach V, etc.)



Velocity

as derived from the upper diagram (using Origin)

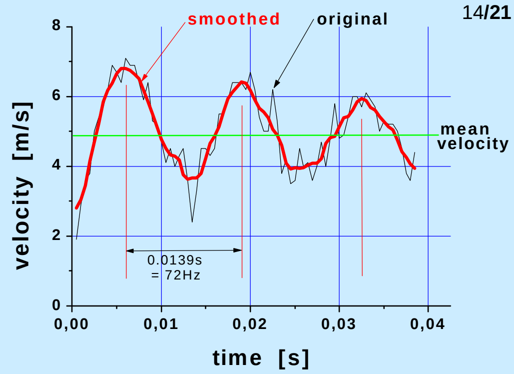


By video analysis, the position of the toy's head is extracted and shown in the upper figure. This seems to be like a badly drawn straight line. But it contains a lot of information. If you analyze this, you get the bottom figure. It shows the velocity as a function of time. The maximum velocity of the head is about $7ms^{-1}$. This is in reasonable agreement with the previously calculated value of $8ms^{-1}$.

Also you can see the oscillation time of the toy (0.0139s) here and calculate the frequency (72Hz).

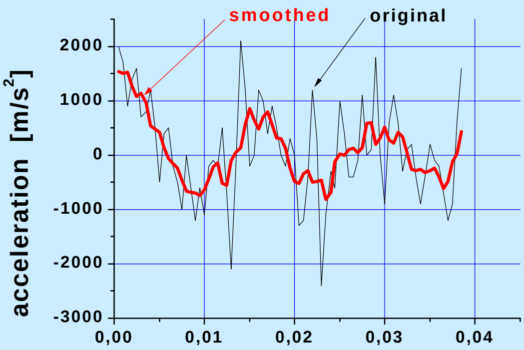
jumping toy

Velocity



Acceleration

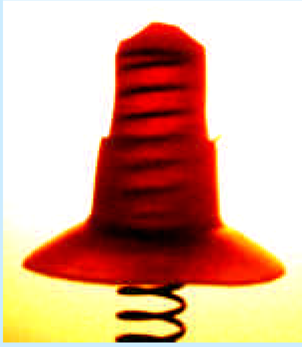
as derived from the upper diagram for the velocity



In the upper figure the velocity of the head is shown again as a function of time. The bottom figure is derived from that and shows the acceleration. The data are smoothed because the double derivation leads to great fluctuations. The measured initial acceleration of about 2000m/s^2 (= 200g) is roughly equal to the calculated value of 190g.

One has to be careful with smoothing. Especially the data near t = 0 are corrupted by this procedure.

jumping toy



Mechanical energy is lost in the suction cup.

Several coils of the spring are compressed into the suction cup;
this part of the spring can decompress only with considerable friction.

Furthermore, there are unpredictable rotations and somersaults which also need energy.

These losses are difficult to calculate quantitatively.

There is also a small fraction of the energy stored in the oscillation between head and base, which can be calculated.

The coils of the spring that are pressed into the top of the suction cup can only move with considerable friction. Furthermore, there are unpredictable rotations and somersaults around several axes which are hard to predict. These cost energy, thus reducing the height. These influences are almost impossible to measure quantitatively.

jumping toy

Using calculus you can write down the differential equation for the starting process. This is the well-known equation for an oscillating mass hanging on a spring (harmonic oscillator), where damping is neglected:

$$\boxed{m_1 \ddot{y} = m_1 g - cy}$$

The solution with the initial conditions $\dot{y}(0) = 0$ and y(0) = -d is

$$y(t) = \frac{m_1 g}{c} + \left(d - \frac{m_1 g}{c} \right) \cos\left(\sqrt{\frac{c}{m_1}} t \right)$$

and

$$\dot{y}(t) = \sqrt{\frac{c}{m_1}} \left(d - \frac{m_1 g}{c} \right) \sin\left(\sqrt{\frac{c}{m_1}} t \right)$$

$$\ddot{y}(t) = \frac{c}{m_1} \left(d - \frac{m_1 g}{c} \right) \cos\left(\sqrt{\frac{c}{m_1}} t \right)$$

Another level is the use of calculus. First year physics students can do this.

You can write down the differential equation for the starting process. This is the well-known equation for an oscillating mass hanging on a spring (harmonic oscillator). Damping is disregarded here but can also be introduced.

By derivation you get the velocity and the acceleration.

If the base is attached to the floor, this equation means that the head oscillates with the amplitude of ($d – m_1g/c$) and with the angular frequency of $1/2pi \cdot sqrt(c/m_1)$ = 39Hz. This is not the same oscillation frequency as previously calculated between head and base!

jumping toy

$$\dot{y}(t) = \sqrt{\frac{c}{m_1}}\left(d - \frac{m_1 g}{c}\right)\sin\left(\sqrt{\frac{c}{m_1}}t\right)$$

For $t = \frac{p}{2}\sqrt{\frac{m_1}{c}}$ $(= 0.0062s = 6.2ms)$ you get the maximum velocity.

The time is not the same as previously calculated, when uniform acceleration was assumed.

The velocity itself is – of course - exactly the same as previously calculated without using calculus.

From the equation for the velocity the time for the starting process and the maximum velocity of the toy can be deduced.

The function has ist maximum for this expression where the sinus has ist maximum.

The velocity is, of course, the same as previously calculated.

Gerace/Dufresne/Leonard
Physics Teacher/Febr. 2001

This model assumes a damping force that is proportional to the relative velocity of the two springbok masses. This model does not attempt to take into account sliding friction. To obtain a "simple" expression for the time at which the springbok leaves the table, we assume that the damping coefficient is small.

Phase 2 simplified equations of motion:

$$\mu\ddot{y} = -b\dot{y} + k(L - y)$$

$$M\ddot{Y} = -Mg$$

Phase 2 solution:

$$y(t) = L + Ae^{-\delta t'}\cos(\lambda t' - \theta) + \frac{v_0}{\lambda}e^{-\delta t'}\sin\lambda t'$$

$$Y(t) = Y_0 + V_0 t' - \frac{1}{2}g(t')^2$$

where:

$$A = \frac{m_t g}{k} + \frac{2}{k}\left\{-Mg\sin^2\varphi + \tan\varphi\left[(kD - m_u g)^2 - (Mg\cos\varphi)^2\right]^{1/2}\right\}$$

$$\delta = \frac{b}{2\mu} \qquad \tan\theta = \frac{\delta}{\lambda}$$

$$\lambda = \sqrt{\beta^2 - \delta^2} \qquad \sin\theta = \frac{\delta}{\beta}$$

$$\beta = \sqrt{\frac{k}{\mu}} \qquad \cos\theta = \frac{\lambda}{\beta}$$

$$v_0 = \frac{\alpha}{k}\left\{Mg\tan\varphi + \left[(kD - m_u g)^2 - (Mg\cos\varphi)^2\right]^{1/2}\right\}$$

$$Y_0 = \frac{m_a}{M}(L + A) \qquad V_0 = \frac{m_u}{M}v_0$$

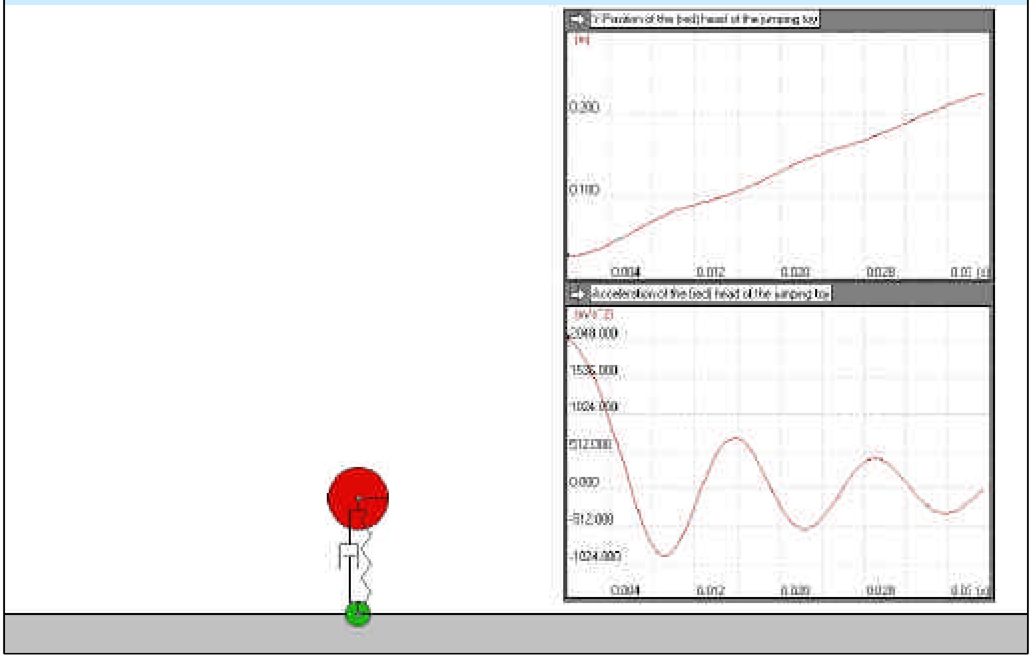
$$t' = t - \frac{1}{\omega}\left\{\cos^{-1}\left|\frac{Mg\cos\varphi}{m_u g - kD}\right| - \varphi\right\}$$

The American scientists Gerace, Dufresne & Leonard have investigated even more accurately the problem of two masses connected by a spring. They call their design a springbok. They don‘t take into account the mass of the spring.

I do not expect that you can read or understand this. You can see that there can be involved much more mathematics.

Dufresne, R.J. et al.: Springbok: The Physics Jumping, The Physics Teacher 39 (2001), 109-115.

jumping toy
Modelling and Simulation with Interactive Physics



A completely another level is to model and simulate the toy on a computer.

There are several programs which allow the simulation of mechanical situations as, for example, the jumping toy. Here I have used the program ‚Interactive Physics‘. It is possible to create and vary almost all parameters of the toy as masses of the head and the base with the spring constant including damping.

It is interesting to see that the results of the simulation are quantitatively very similar to the real experiment, as you perhaps remember from the previously shown graphs.

The simulation is very informative because you can easily vary all the parameters.

You can model easily your toy with Interactive Physics (cost of the program about 200 USD)

There are not many toys which have that many advantages :

1) **Cheap**
2) **Interesting and motivating for children**
3) **Simple and transparent design**
4) **Easy to build by yourself**
5) **Interdisciplinary reflections**
6) **Comparison experiment – theory**
7) **Modelling and Simulation**
8) **Different levels**

But nothing is without disadvantage:

1) **Certain danger**
2) **Not always available**

At the end let me express that there are not many toys which have that many advantages.

GIREP Seminar 2001 Udine