

Playing with caustic phenomena

Christian Ucke/Christoph Engelhardt

Techn. Univ. Munich, Physikdept. E20, D-85747 Garching, Germany

Abstract: On a sunny day in a well-filled cup of coffee one often can observe a caustic. This is the well-known curve that is produced by the reflection of rays of light in a circular mirror. The phenomena one observes depend on the angle of incidence and the height of the mirror. They can be described mathematically with epicycloids. Not so well-known are the caustics which result from multiple reflections. They can be easily obtained with circular cylindrical mirrors producing aesthetic epicycloids in the form of a heart. I will show how to create these caustics. The mathematical derivation is given. The graphical representation is possible in a few lines with mathematical program packets like MATHEMATICA.

On a sunny day one can often observe a caustic in a well-filled cup of coffee. This is the well-known curve produced by the reflection of light-rays in a circular mirror. I have inserted a mirror sheet in this coffee cup so that the reflection is much better than in regular cups. You need a little bit of milk in the coffee. By the way, tea drinkers can of course produce the same phenomenon.



Fig.1: The rays of the sun are reflected from the inner part of a coffee cup and form a caustic on the surface of the coffee.

When the sun sets - this means the angle of incidence changes - the shape of the caustic also changes. The sharp contour will be softer. An additional, very faint curve appears. The same effect will occur when you drink some coffee changing the level of the coffee. This way, you can easily observe and enjoy the phenomenon, because waiting for a sun set will take too long.

This effect can be seen better in a well-polished, circular cylindrical wedding ring. The whole phenomenon can be studied even better in a bigger cylindrical mirror, which is polished on the inner side and which can be made out of a brass cylinder in a workshop.

About mid 1995, a physics student (C.E.) asked me (C.U.) whether I knew the phenomenon of the caustic in a coffee cup. This student is the co-author of this paper. I said

yes, and he was a little disappointed when I told him that this has been known for several hundred years and that the Dutch physicist Christian Huyghens (1629-1695) had described them and that, in the seventeenth century, the Swiss mathematician Johann Bernoulli (1667-1748) had already described the form of the caustic mathematically as a special form of a cycloid, a so-called epicycloid [1]. A cycloid is a curve generated by a point carried by a curve which rolls on a fixed second curve. In the particular case in which the point is on the circumference of the rolling circle and the circle rolls on the outside of another circle, the corresponding curves are epicycloids (epitrochoids).

But then I asked him whether he had observed the effect of multiple reflections in a cylinder. By chance I had polished, circular cylinders on me and could show him the following pictures.

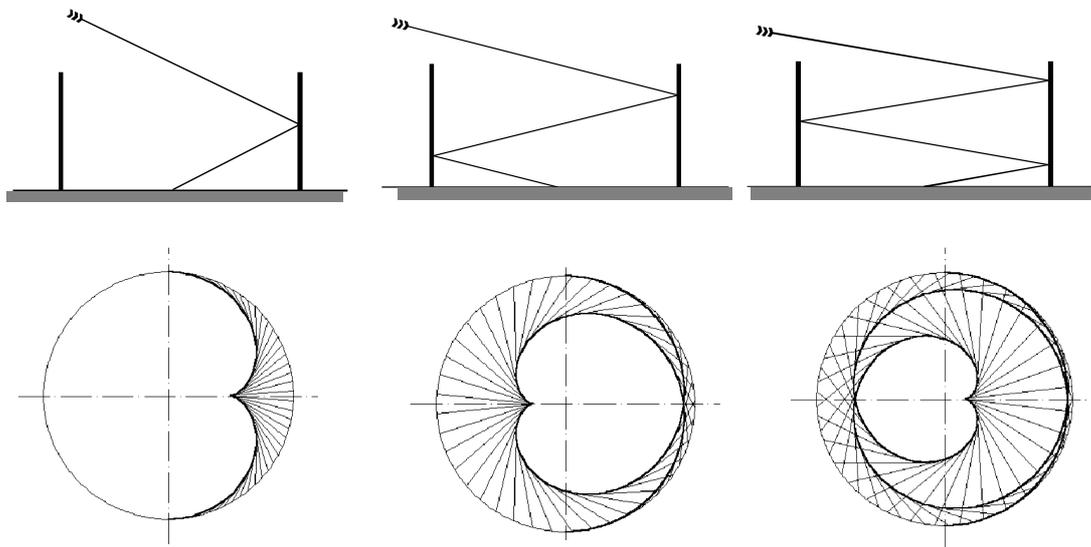


Fig.2: In the left figure there is the regular, well-known 'classical' caustic, which appears on the bottom of the cylinder, if the the rays are reflected one time. Only the reflected rays of the order n are drawn. In the middle the rays are reflected two times, as indicated schematically above - a heart-like curve appears. On the right side the rays are reflected three times. Again a heart appears. These pictures are drawn with a MATHEMATICA-program. Real photographs would be too difficult to reproduce with the printing technique used in these proceedings.

In the first picture you see the regular caustic. In the second picture there are already two reflections - and suddenly a curve shaped like a heart appears. In the third picture there are three reflections. And again a heart appears but reversed from the first one. And so on. In polished brass cylinders you can hardly see more than the fifth or sixth reflection. In circular wedding rings you can see a maximum of two or three reflections.

But isn't it nice that hearts appear in wedding rings? Because of this heart-like shape the epicycloids are also called cardioides.

My co-author was so interested in this that he started to derive the mathematical description of these multiple reflection curves. There is not enough space here to develop this in detail. I can only outline a short sketch of how to achieve these results.

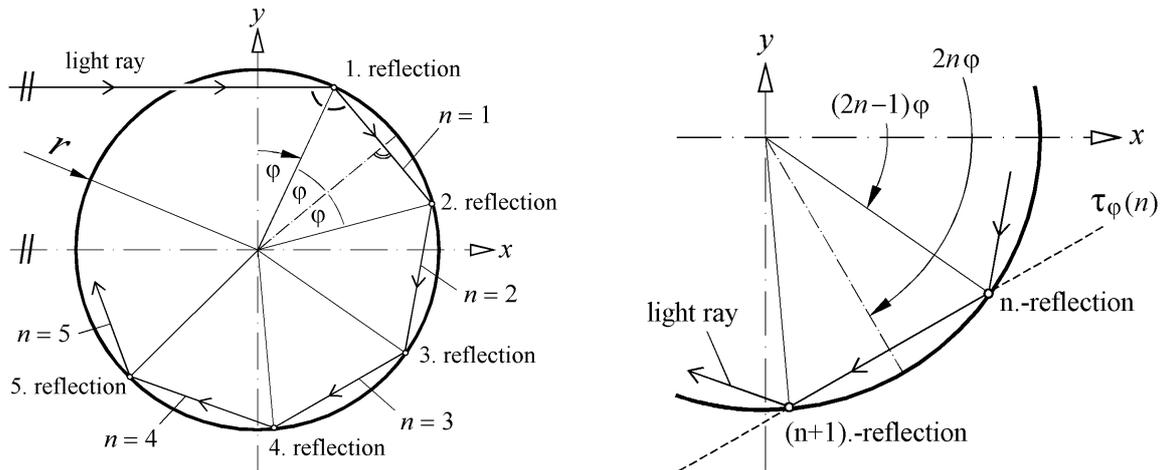


Fig.3: On the left side a cylinder is drawn with a light ray reflected several times in it. On the right side you have the light ray for the n-th reflection. r is the radius of the cylinder.

The equation for the straight line is

$$t_j(n): x \sin(2n\mathbf{j}) + y \cos(2n\mathbf{j}) - r \cos \mathbf{j} = 0$$

If you derive this equation partially, you get

$$\frac{\partial}{\partial \mathbf{j}} t_j(n): 2nx \cos(2n\mathbf{j}) - 2ny \sin(2n\mathbf{j}) - r \sin \mathbf{j} = 0$$

If you solve both of these equations for x and y, you get the equations for the caustic that expresses the envelope of all the straight lines mathematically. Here are the equations which represent cycloids, more exactly epicycloids.

$$x(\mathbf{j}) = \frac{r}{4n} [(2n+1) \sin((2n-1)\mathbf{j}) + (2n-1) \sin(2n+1)\mathbf{j}]$$

$$y(\mathbf{j}) = \frac{r}{4n} [(2n+1) \cos((2n-1)\mathbf{j}) + (2n-1) \cos(2n+1)\mathbf{j}]$$

When $n = 2$, this is the caustic of second order reflection, you get the epicycloid shown in figure 4. This results from one circle K rolling on another circle L and the rolling circle K has a diameter twice the size of the fixed one L. If you take a point on the rolling circle and follow it, you will get the epicycloid.

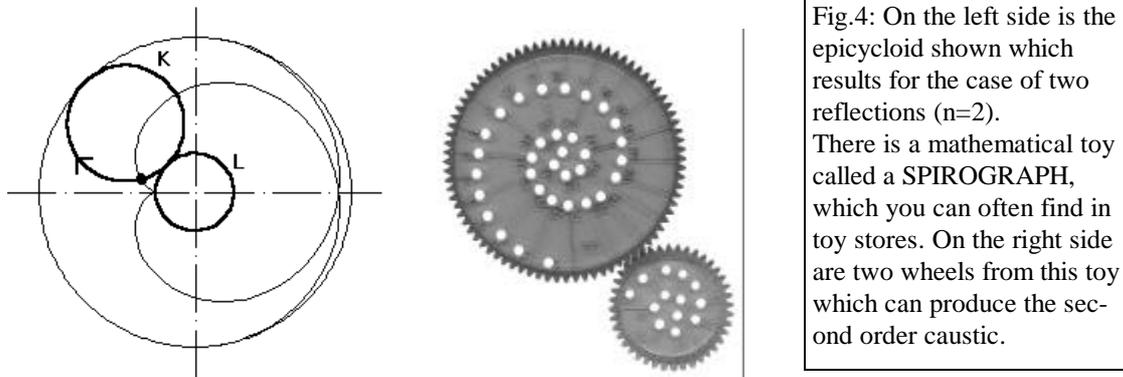


Fig.4: On the left side is the epicycloid shown which results for the case of two reflections ($n=2$). There is a mathematical toy called a SPIROGRAPH, which you can often find in toy stores. On the right side are two wheels from this toy which can produce the second order caustic.

Nowadays with MATHEMATICA you have a powerful program packet to show the graphs of curves such as these epicycloids. The program code is very short:

MATHEMATICA-PROGRAM

```
(* n is the order of reflections, k the number of the light rays to be drawn*)
(* t is equivalent to  $\phi$  ; compare figure 3 *)
(* caustic[n_] calculates the caustic of the order n *)
(* lightrays[n_,k_] calculates k reflected light rays of the reflection order n *)

n = 3; k = 201;

xk[n_,t_] := 1/(4n) ((2n + 1) Sin[(2n - 1)t] + (2n - 1) Sin[(2n + 1)t]) ;
yk[n_,t_] := 1/(4n) ((2n + 1) Cos[(2n - 1)t] + (2n - 1) Cos[(2n + 1)t]) ;

caustic[n_] := ParametricPlot[ { xk[n,t] , yk[n,t] } , {t,0,Pi} ,
    DisplayFunction -> Identity , PlotStyle -> AbsoluteThickness[.55]] ;

lightrays[n_,k_] := Graphics[ { Thickness[.0015] , Evaluate[ Table[ Line[
{ { Sin[(2n - 1)t] , Cos[(2n - 1)t] } , { xk[n,t] , yk[n,t] } } ] , {t,0,Pi,Pi/k} ] ] ] ;

cylinder := Graphics[ { AbsoluteThickness[.65] , Circle[ {0,0} , 1 ] ] ] ;

Show[ { caustic[n] , lightrays[n, k] , cylinder } ,
    DisplayFunction -> $DisplayFunction , AspectRatio -> Automatic ,
    PlotRange -> {{-1.2,1.2},{-1.2,1.2}} ,
    Ticks -> {{-1,-.5,.5,1},{-1,-.5,.5,1}} , Axes -> False ] ;
```

When C.E. had made all these calculations and graphic representations, we discovered a publication by the English author H. Holditch [2]. He described the same matter in a somewhat old-fashioned way, which is difficult for us to understand.

Although we have looked through existing databases thoroughly, this publication could not be found. It's too old. We got the hint from an author of other publications about caustics (Peter Giblin from the University of Liverpool), who is still alive. Ask experts!

The simple-looking first order caustic that it is shown in the beginning (fig.1) is more complicated to calculate. Several cases must be distinguished. The shape of the caustic depends on the angle of incidence and/or the height h of the cylinder. In figure 5, different cases are outlined.

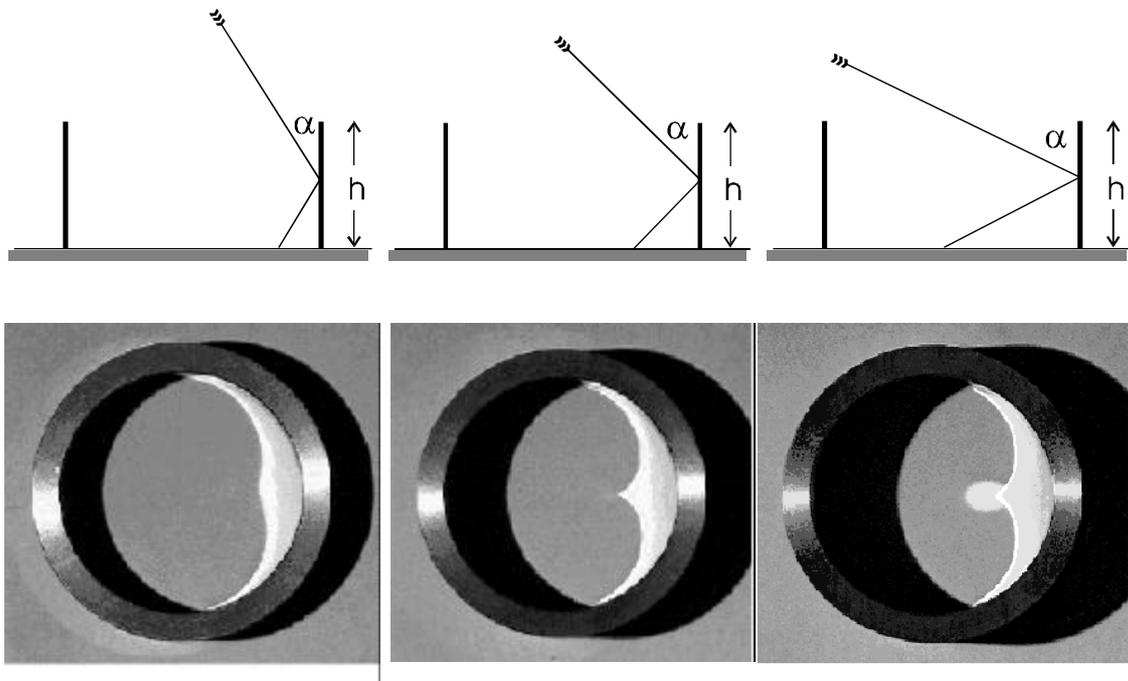


Fig.5: The shape of the caustic of first order reflection depends on the angle of incidence α and/or the height h of the cylinder. In the left picture, $h \cdot \tan \alpha < r/2$ (r = radius of the cylinder). In the centre picture, $h \cdot \tan \alpha = r/2$; this is the 'classical caustic' with a sharp cusp. In the right picture, $h \cdot \tan \alpha > r/2$. You can see the entire classical caustic and an additional loop. The photographs were made with a polished brass cylinder and an artificial point light source.

All these shapes are cycloids or combinations of cycloids. In the case of a large incidence angle α (right picture in fig.5) the shape is a combination of an epicycloid and a so-called prolate epicycloid, as it is shown in fig. 6.

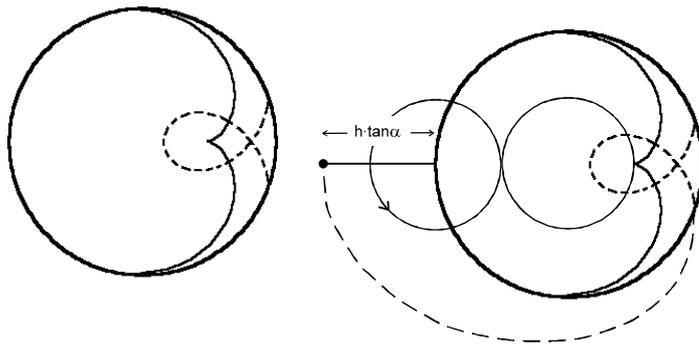


Fig.6: In the case of a large incidence angle α the shape of the light appearance consists of two epicycloids (left side): the 'classical caustic' as an epicycloid, for which the rolling circle has half the diameter of the fixed circle. And a loop (dotted curve) as a prolate epicycloid. This means the point of the rolling circle does not lie on the circumference but on a radius outside the circumference (right side).

The loop, i. e. the prolate epicycloid, can be described by the following parametric equations:

$$x(\mathbf{j}) = r \cdot \sin \mathbf{j} + h \cdot \tan \mathbf{a} \cdot \cos 2\mathbf{j}$$

$$y(\mathbf{j}) = r \cdot \cos \mathbf{j} - h \cdot \tan \mathbf{a} \cdot \sin 2\mathbf{j}$$

$h \cdot \tan \alpha$ gives the distance of the moving point from the center of the rolling circle.

What we have described here is of course a very short description of the way we have gone. When we had calculated one curve and compared it with reality we made new observations. For instance there are more problems to discover in the corner parts of the caustics. I believe that this was a typical way of problem solving in physics. The mathematics of this case, the first order caustic, will be published in a separate paper [3].

Without MATHEMATICA it would be hard to produce the graphics. Using MATHEMATICA, C.E. has written a whole animation of the caustic depending on the angle of incidence [3].

- [1] **Joh. Bernoulli:** Lectiones mathematicae de methodo integralium aliisque conscriptae in usum ill. Marchionis Hospitalii, Opera omnia, t. III, p. 386-558, Paris 1692
- [2] **H. Holditch:** On the n-th caustic, by reflexion from a circle, Quarterly Journal of Mathematics 2 (1858), 301-322
- [3] **C. Engelhardt:** to be published

This whole article with some additional color photographs of caustics can be downloaded by ftp from: <ftp.e20.physik.tu-muenchen.de/pub/caustic/>